We know addition undoes subtraction and square rooting undoes squaring. What undoes taking the derivative?

Business Calculus
Class notes
Section 4.1: Antidifferentiation

When we differentiate a function like $g(x) = x^2$, we get $g'(x) = 2x$.

But can we go the other way? What if we were given $f(x) = 2x$ and asked to find the function whose derivative would be this $f(x)$? Would there be just one possible answer?

This is the idea of the antiderivative. A calculus book presents it in this traditional way.

**DEFINITION Antiderivative**

A function $F$ is an **antiderivative** of $f$ on an interval $I$ provided $F'(x) = f(x)$, for all $x$ in $I$.

Notice that we use capital $F$ and lower case $f$ so be careful with that. If you are *not* already in the habit of maintaining case with variables, please work on that. For the example we started with, we say that $F(x) = x^2$ is the **antiderivative** of $f(x) = 2x$ because $F'(x) = f(x)$.

So, we will be given a function which is said to be the derivative of an unknown function. It will be our job to figure out this unknown function. This concept will lead us to some truly lovely mathematics called **integration**. But, first things first …

Let me give you another one. Suppose that $f(x) = 3x^2$. Can you guess its antiderivative? Or rather, *can you guess the function whose derivative is this* $f(x) = 3x^2$?

$H(x) = x^3 + 4$ } $F(x) = x^3$ (because $F'(x) = 3x^2 = f(x)$

If you said $F(x) = x^3$, then I say "Bravo". But is that the only possible answer? Can you think of another function whose derivative is $f(x) = 3x^2$? When you come up with others, name them H or G or the like. To be clear, any function whose derivative is $f(x) = 3x^2$ is considered an **antiderivative** of $f(x)$. We call this a family of antiderivatives.

**THEOREM 4.14 The Family of Antiderivatives**

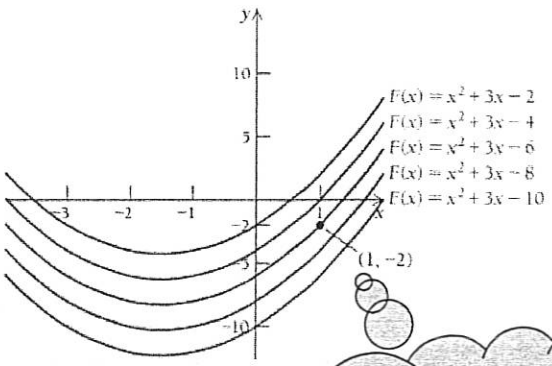
Let $F$ be any antiderivative of $f$ on an interval $I$. Then *all* the antiderivatives of $f$ on $I$ have the form $F + C$, where $C$ is an arbitrary constant.

This $C$ is the **constant of integration**.

$F'(x) = f(x)$

The book gives a great example of this.

Consider some function $f(x) = 2x + 3$ whose antiderivative is $F(x)$. In other words, $\frac{d}{dx}F(x) = f(x)$. Use derivative rules to see that $\frac{d}{dx}\left(x^2 + 3x + C\right) = 2x + 3$. That leads us to know that $F(x) = x^2 + 3x + C$. $C \in \mathbb{R}$




Notice $C$ could be *any* constant as its derivative would always be 0. Below we will introduce the notation $\int (2x+3)dx = x^2 + 3x + C$.

If given a point on $F(x)$, we can narrow down $C$ to a single value.

When you add or subtract a real number from a function (these values of $C$), it graphs as a vertical translation.

**Integration Notation:**

For this new concept, we have new notation. We will write $\int f(x)dx = F(x)$ or, equivalently, $\int F'(x)dx = F(x)$. That symbol can be thought of as an elongated S for "sum"; that will be explained later. We say this is the **integral** of $f(x)$ with respect to $x$.

You can think of the $\int$ symbol and the $dx$ part as two pieces of the same symbol. They say to find the antiderivative of the function $f(x)$ taking $x$ as the **independent variable**. This function $f(x)$ is called the **integrand**.

We would write $\int 2x dx = x^2 + c,\ c \in \mathbb{R}$ since $\frac{d}{dx}\left(x^2 + c\right) = 2x$. We are undoing the act of taking a derivative, just as we learned square rooting undoes the act of squaring a number.

Do *not* leave off the $dx$ part when writing integrals. It will be more meaningful as we add complexity.

Just as we had many shortcut rules to help us find derivatives, we have integration rules.

Look carefully and you'll see that they are simply undoing the very process we practiced with differentiation.

So, put your shoes on backwards and grab a mirror. Backwards, ho!

Recall, power rule for derivatives

$$\frac{d}{dx}(x^n) = n \cdot x^{n-1}$$

**THEOREM 4.15 Power Rule for Indefinite Integrals**

$$\int x^p \, dx = \frac{x^{p+1}}{p+1} + C,$$

where $p \neq -1$ is a real number and $C$ is an arbitrary constant.

Why would this *not* work if $p = -1$? (division by 0)

**THEOREM 4.16 Constant Multiple and Sum Rules**

**Constant Multiple Rule:** $\int cf(x) \, dx = c \int f(x) \, dx$, for real numbers $c$

**Sum Rule:** $\int (f(x) \pm g(x)) \, dx = \int f(x) \, dx \pm \int g(x) \, dx$

For instance, $\int 2x dx = 2\int x dx$.

This works for subtraction too.

I would like to add a useful $\int 1 dx = \int x^0 dx = x^1 + c, \; c \in \mathbb{R}$. This is a direct result of the top rule but it's nice to explicitly state. I will also add $\int \frac{1}{x} dx = \int x^{-1} dx = \ln|x| + c, c \in \mathbb{R}$ as it fills the gap in the first rule. Let's add $\int e^{ax} dx = \frac{1}{a} e^{ax} + C, \quad a \neq 0$. (Some sources use $c$ while others use $C$.)

Recall, $\frac{d}{dx}(\ln x) = \frac{1}{x}$

We'll also use $\int a^x dx = \frac{1}{\ln a} a^x + C$ and $\int \frac{1}{x \ln a} dx = \log_a x + C$ where $x \neq 0$, $a \neq 1$, and $a > 0$.

These are called *indefinite* integrals. When we see the graphical meaning of a function's integral, this will make more sense. We will study both definite and indefinite integrals. For now, think of them as undoing the derivative process.

Integration answers some interesting questions. For example, if we know a company's marginal profit, $P'(x)$, after $x$ units have been sold, we can integrate to find the company's total profit after $x$ units have been sold.

Or, given the function $f'(t)$ which is the rate of change in the spread of a virus $t$ weeks after case 0, we can integrate it to find the total number of people infected after $t$ weeks.

We will need a good deal of algebra too. Recall the factoring formulas below.

$a^2 - b^2 = (a + b)(a - b)$

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

The sum and differences of cubes and squares will play a role in simplifying before integration.

SOAP anyone?

We need to find *all* antiderivatives. So, include $c \in \mathbb{R}$ in your answers.

example 1: Determine the integral.

$\int \left( \frac{5}{t^2} + 4t^2 \right) dt$

$= \int (5t^{-2} + 4t^2)\, dt$

$= 5\frac{t^{-1}}{-1} + 4\frac{t^3}{3} + C$

$= \boxed{\frac{-5}{t} + \frac{4}{3}t^3 + C, \; C \in \mathbb{R}}$

$\int x^p dx = \frac{x^{p+1}}{p+1} + C$

As when finding derivatives, I like to rewrite terms as non-fractions and without radicals.

example 2: Determine the integral.

$\int \left( 6\sqrt[3]{w} + 4^w \right) dw$

$= \int (6w^{1/3} + 4^w)\, dw$

$= \frac{6}{1}\cdot\frac{3}{4_2} w^{4/3} + \frac{1}{\ln 4}\cdot 4^w + C$

$= \boxed{\frac{9}{2} w^{4/3} + \frac{4^w}{\ln 4} + C, \; C \in \mathbb{R}}$

$\int a^x dx = \frac{1}{\ln a} a^x + C$ (pg 3)

$\frac{1}{3} + 1 = \frac{1}{3} + \frac{3}{3} = 4/3$

You can always check by finding your answer's derivative.

We do *not* have a rule for the integral of a quotient. However, algebra can reduce this.

example 3: Determine the integral.

$\int \frac{y^3 - 64}{y - 4} dy$

$= \int \frac{(y-4)(y^2+4y+16)}{y-4}\, dy$

$= \int (y^2 + 4y + 16)\, dy$

$= \frac{y^3}{3} + \frac{4y^2}{2} + 16y + C$

$= \boxed{\frac{y^3}{3} + 2y^2 + 16y + C}$

SOAP

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$y^3 - 4^3 = (y-4)(y^2 + 4y + 4^2)$

$\int x^p dx = \frac{x^{p+1}}{p+1} + C$

$\int 1\, dx = x^1 + C$

Go term by term.

**Recall: Differential Equations:**

You will be given an equation that involves a derivative of an unknown function and asked to identify the function. This sounds like what we have been doing, doesn't it? As we have seen, that function would have to be written with some constant in place ( $c \in \mathbb{R}$ ). Often, we will be given more information so that we can solve for that constant as well. These are called **initial value problems**.

We are asked for $g(x)$ where $g(x) = \int g'(x)dx$. Don't forget the constant.

example 4: Find the solution to the initial value problem.

$g'(x) = 7x^6 - 4x^3 + 12$; $g(1) = 24$

$g(x) = \int g'(x)dx$

$= \int (7x^6 - 4x^3 + 12)\,dx$

$g(x) = \frac{7x^7}{7} - \frac{4x^4}{4} + 12x + C$

$\boxed{g(x) = x^7 - x^4 + 12x + C}$

$g(x) = x^7 - x^4 + 12x + C$

$24 = 1^7 - 1^4 + 12(1) + C$

$24 = 1 - 1 + 12 + C$

$\boxed{C = 12}$

So, soln is $g(x) = x^7 - x^4 + 12x + 12$

**Applications in General:**

Whatever the function involved, if we know its rate of change (derivative), then we can use integration to find the function down to a constant. Further if we are given some point on this function (an initial point or condition), then we can solve for that constant and know the function exactly.

$P(0) = 2,811,400$

example 5: The population of the greater St. Louis metropolitan area was 2,811,400 in 2010. Since then, our population has grown at a small but positive rate such that $\boxed{P'(t) = 196.798 \cdot e^{0.000\,07t}}$ is the population's rate of growth in people per year. Find the function that gives our area's population $t$ years after 2010.

$\int e^{ax}dx = \frac{1}{a}e^{ax} + C$

$a = 0.000\,07$

$P(t) = \int P'(t)\,dt$

$= \int 196.798\, e^{0.000\,07t}\,dt$

$= 196.798 \cdot \frac{1}{0.000\,07} \cdot e^{0.000\,07t} + C$

$P(t) = 2,811,400\, e^{0.000\,07t} + C$

$2811400 = 2811400\, e^{0.000\,07(0)} + C$

$C = 0$

$\boxed{P(t) = 2,811,400\, e^{0.000\,07t}}$

$\int x^p dx = \frac{x^{p+1}}{p+1}$

example 6: Paul's Pirated Parachutes determines that the marginal profit, $P'(x)$, in hundreds of dollars per unit, from selling the $x^{th}$ parachute is given by $P'(x) = 0.02x + 1.45$. It is also known that the profit is \$0 when 0 parachutes are sold. (In other words, $P(0) = 0$.)

a.) Find the profit function, $P(x)$.

We know $P(x) = \int P'(x)dx$.

$P(x) = \int P'(x)\,dx$

$= \int (0.02x^1 - 1.45)\,dx$

$P(x) = \frac{0.02x^2}{2} - 1.45x + C$

$P(x) = 0.01x^2 - 1.45x + C$

$P(0) = 0$

$0 = 0.01(0)^2 - 1.45(0) + C$

$0 = C$

$P(x) = 0.01x^2 - 1.45x$

b.) How many parachutes do they need to sell to break even?

What does it mean to "break even"?

"make back what you originally spent"

OR $R(x) = C(x)$

OR $P(x) = 0$

$P(x) = 0.01x^2 - 1.45x$

$0 = 0.01x^2 - 1.45x$

$0 = x(0.01x - 1.45)$

$x = 0$ or $0.01x - 1.45 = 0$

ignore

$0.01x = 1.45$

$x = 145$ parachutes

Sell 145 parachutes and you'll break even.