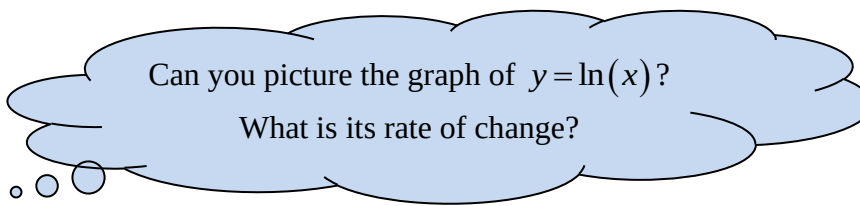


Section 2.3: Derivatives of Natural Logarithmic Functions



The natural logarithmic function (usually written as $y = \ln(x)$ which means $y = \log_e(x)$) may be involved in problems where we want to know the function's rate of change. So, what is the derivative of $y = \ln(x)$? The book has a quick proof of this but we just give the rule here.

Derivatives of Natural Logarithmic Functions:

For any positive number x (because the function's *not* defined otherwise), we have

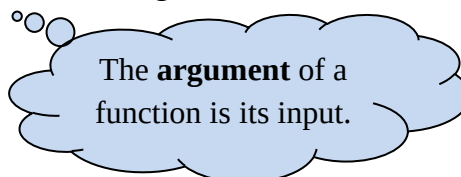
$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

But what if the thing we're taking the natural log of is *not* just x , but rather a function in x ? Then, we'd need the Chain Rule. So, we have this lovely nugget.

$$\frac{d}{dx}(\ln f(x)) = \frac{1}{f(x)} \cdot f'(x) = \frac{f'(x)}{f(x)}$$

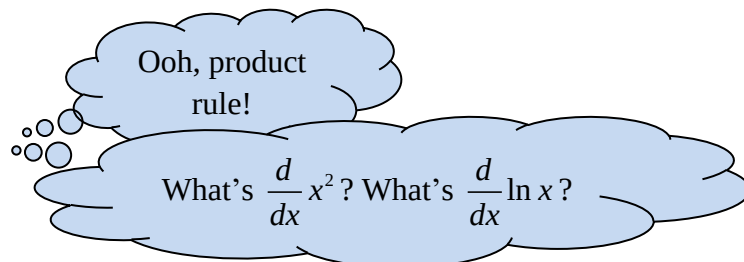
Equivalently, if we think of the argument of the log function as u which is a function in x , we would write this.

$$\frac{d}{dx} \ln u = \frac{1}{u} \cdot \frac{du}{dx}$$



Let's do some examples.

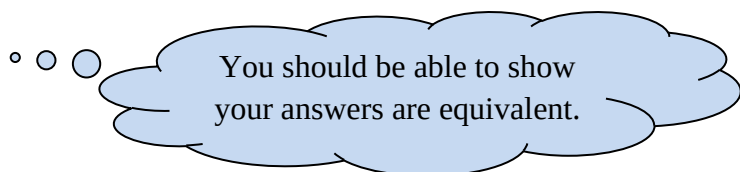
expl 1: Find the derivative of $y = x^2 \ln x + 4$.



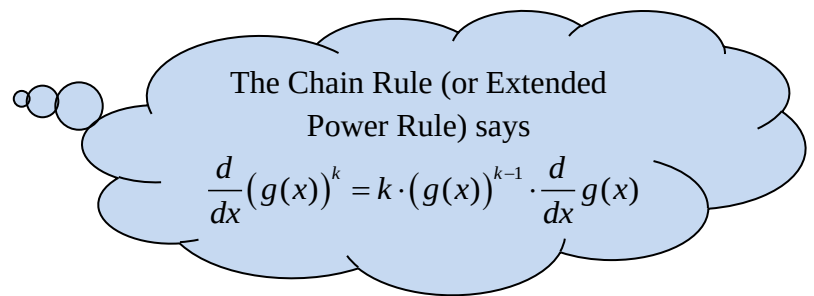
2. Find the derivative of $y = \ln\left(\frac{x+4}{x^2-8}\right)$. We will do this using two different methods to explore our options.

a.) Method 1: Quotient rule to find $\frac{d}{dx}\left(\frac{x+4}{x^2-8}\right)$ because $\frac{d}{dx}\ln u = \frac{1}{u} \cdot \frac{du}{dx}$

b.) Method 2: Logarithm Properties (specifically, $\ln\left(\frac{M}{N}\right) = \ln M - \ln N$) to avoid Quotient Rule



3. Find the derivative of $f(x) = (\ln x)^5$.



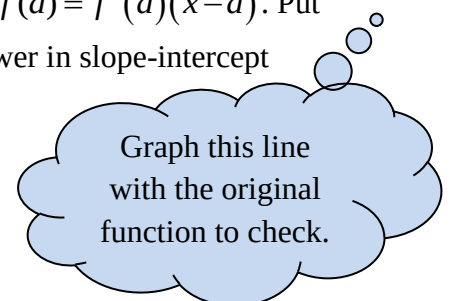
4. Find the equation of the tangent line to the graph of $f(x) = (\ln x)^5$ at $x = 4$. Follow these steps.

a.) Graph the function and copy it here.

The Standard window will be fine. Use the Value function on your calculator to find $f(4)$ to four decimal places. On your paper, plot and label the point at $x = 4$ and sketch the tangent line.

b.) Find the slope of this tangent line at this point. In other words, find $f'(4)$. Round to four decimal places.

c.) Recall that the equation of this tangent line (at $x = a$) is always $y - f(a) = f'(a)(x - a)$. Put in what you know to find the equation of this line. Write your final answer in slope-intercept form.



Recall: Rates of Change, in General:

We are working with specific examples here but understand that these ideas will work for any function. Let's say you know a function $p(t)$ measures something like a population or price per item or number of internet users, etc. Here, t is taken to be time where $t \geq 0$. This function's

instantaneous rate of change is its derivative $p'(t)$ or $\frac{dp}{dt}$. This measures how fast $p(t)$ is changing with respect to time. The units of this rate of change will always be "units of $p(t)$ divided by units of t ".

Application: Marginal Cost, Profit, and Revenue:

Let $C(x)$ be the cost of producing x items. Perhaps we will be given the function $C(x) = -0.02x^2 + 50x + 100$ dollars.

Recall that profit equals revenue minus cost.

The **Marginal Cost** is the approximate cost to produce *one additional item* after producing x items. And that sounds like rate of change. Hence, marginal cost is $C'(x)$.

Further, let $p(x)$ denote the sale price per item if x items are sold.

We would find **Revenue** (which is total money brought in) as the product $R(x) = x \cdot p(x)$.

Notice this is lowercase p .

Which leaves us **Profit** (the money we really made) as

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= x \cdot p(x) - C(x) \end{aligned}$$

Here's the uppercase P .

We are in the US so our units for cost, revenue, and profit will usually be in dollars with the units of their derivatives derived accordingly.

It is *not* a long stretch to define marginal revenue and profit as you see here.

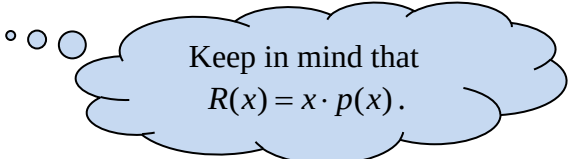
The **Marginal Revenue**, $R'(x)$, is the approximate revenue gained or lost by selling *one additional item* after selling x items.

The **Marginal Profit**, $P'(x)$, is the approximate profit gained or lost by selling *one additional item* after selling x items.

expl 5: The demand for a shirt can be modeled by $p(x) = 65.9 - 10.1 \ln x$, $0 \leq x \leq 680$. Here, $p(x)$ is the price wholesalers will pay, in dollars, and x is the number of shirts that will be sold, in thousands. Answer the following questions.

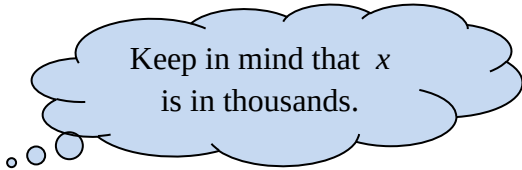
a.) If the price wholesalers are willing to pay is \$20, how many shirts would be sold? Play with solving it algebraically and graphically.

b.) Find a function for the revenue $R(x)$ of this shirt.



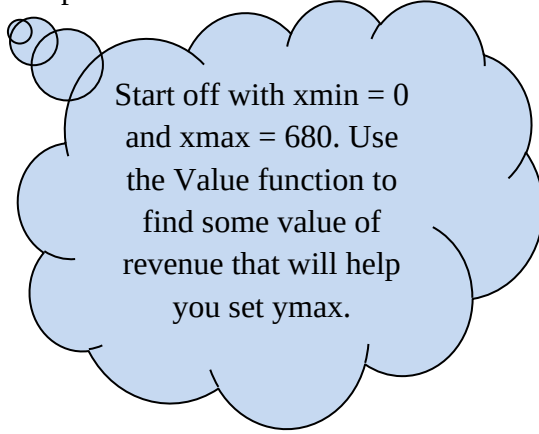
Keep in mind that
 $R(x) = x \cdot p(x)$.

c.) Find a function for the marginal revenue which is $R'(x)$.



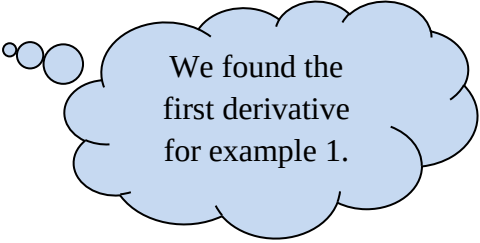
Keep in mind that x
is in thousands.

d.) Find and interpret $R'(94.118)$. Include a graph of $R(x)$ and mark the point where $x = 94.118$.



Start off with $x_{\min} = 0$
and $x_{\max} = 680$. Use
the Value function to
find some value of
revenue that will help
you set $y_{\max}$.

expl 6: Find $\frac{d^2y}{dx^2}$ for the function $y = x^2 \ln x + 4$.



We found the
first derivative
for example 1.