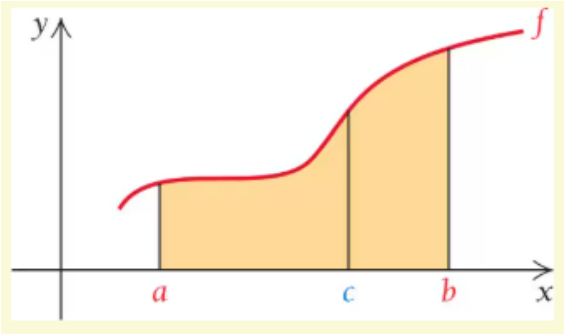
How do you find the area between two functions? What is the average value of a function over an interval?

Business Calculus
Class notes
Section 4.4: Properties of Definite Integrals: Additive Property, Average Value, and Moving Average

The Additive Property of Definite Integrals:
Consider this picture I got from our book. Given that $\int_a^b f(x)dx$ is the shaded area bounded by the x-axis and the curve, how would you find this area if you knew both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$?



This understanding is called the **Additive Property of Definite Integrals** and is quite useful in certain problems.

This is a continuous piecewise function.

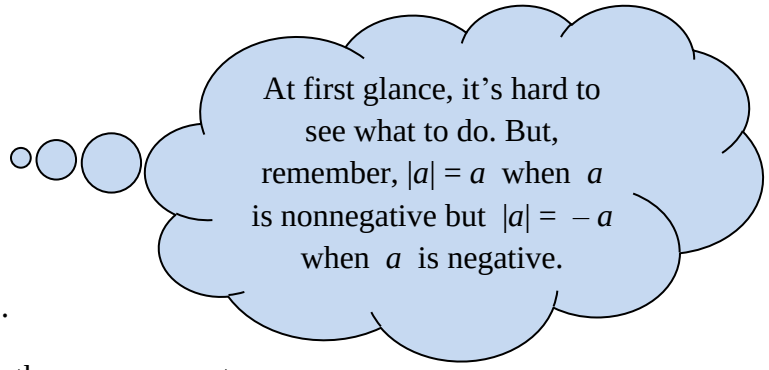
example 1: Find the area under the graph of $g(x)=\begin{cases} x^2+4, & x\le 0 \\ 4-x, & x>0 \end{cases}$ over [-2, 3]. Follow these steps.

a.) Draw a graph. Mark both x = -2 and x = 3, drawing vertical lines and shading the area we are after.

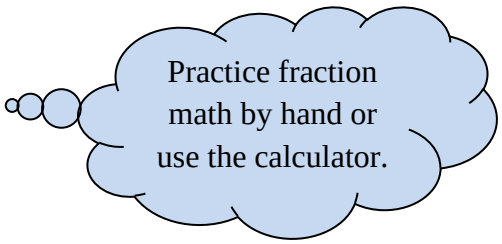
b.) Notice that $\int_{-2}^{3} g(x)dx=\int_{-2}^{0}(x^2+4)dx+\int_{0}^{3}(4-x)dx$.

Find the value.

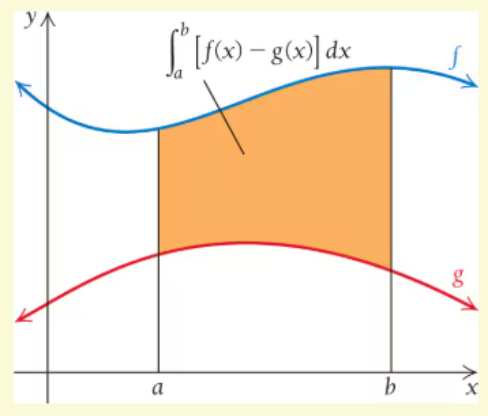
example 2: Evaluate $\int_{-1}^{1} |3x-2|\,dx$. Start with a graph of $y = 3x - 2$ so you can see when $|3x - 2| = 3x - 2$ and when $|3x - 2| = -(3x - 2)$.



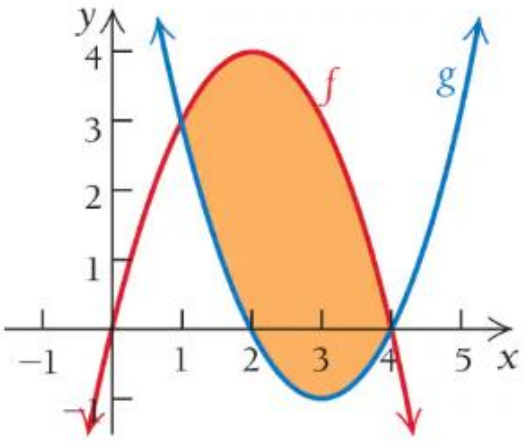
Give yourself a graph of $y = |3x - 2|$ and shade the area we want.



The Area Between Two Functions:
There are many applications when we must find the area between two functions. This picture from our book gives us the important fact that this area can be found by finding the definite integral of the two functions' difference.



example 3: Pictured to the right are $f(x) = 4x - x^2$ and $g(x) = x^2 - 6x + 8$. Find the shaded area between them. Follow these steps.
a.) Algebraically find the x-values where f and g intersect. Verify graphically. These will be the limits of integration.



b.) Use the principle above to find this shaded area.

The Average Value of a Continuous Function:

We can average five exam scores by adding them up and dividing by 5, right?

We can actually do the same thing with function values (or the y-values of a function). If we consider our n subintervals (from Riemann Sums) and call the x-values along the way by our familiar x_i notation, we would write this average as $\frac{f(x_1)+f(x_2)+f(x_3)+\ldots+f(x_n)}{n}$.

With a bit more manipulation, we would end up with what the book has here.

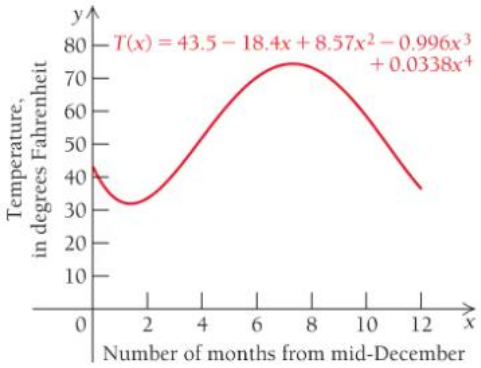
Definition: Average Value:

For f, a continuous function over $[a, b]$, the **average value**, y_{av}, is given by

$$y_{av} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

So, to find the average value of a function over some interval, first find the definite integral and divide by the width of the interval.

example 4: This graph shows the function $T(x) = 43.5 - 18.4x + 8.57x^2 - 0.996x^3 + 0.0338x^4$ where $T(x)$ represents the Fahrenheit temperature in New York City on the 15th day of each month. (December of the previous year is $x = 0$. January is month $x = 1$ and this goes through December which is month $x = 12$.)

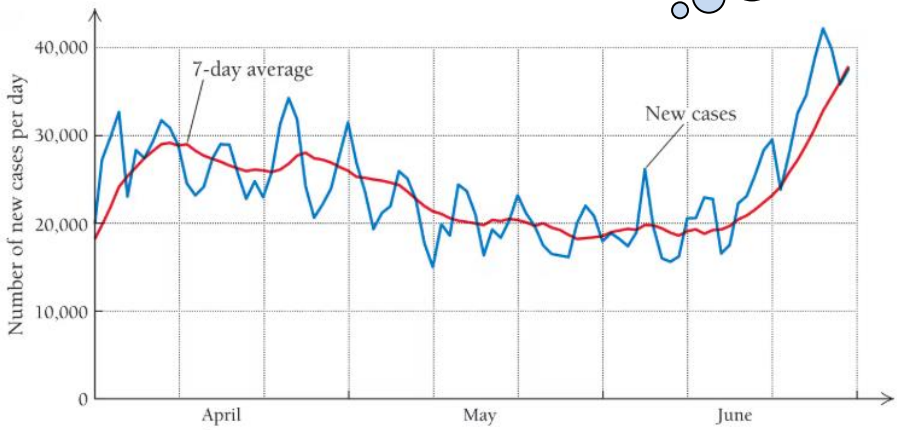


Compute the average temperature for New York City over the whole year to the nearest degree. (Go from December to December.)

The Moving Average:

This is quite an interesting idea. data over time can fluctuate wildly. Below is the graph of Covid cases in the US between April 1 and June 30 of 2020 (jagged line).

This is *not* represented in the homework. I include here because it's a neat idea.



(*Source*: New York Times, CDC)

The wild fluctuations can make finding an average value as seen earlier of little use. Instead, we use a **moving average**. The idea is that you would find averages for every 7 days, instead of the whole two-month period. That is what the red (smoother) line above is doing.

The book gives us this definition.

Definition: Moving Average:

Let f be a continuous function over $[a, b]$. The **moving average function** of f is given by

$$f_{\text{av}}(x) = \frac{1}{L}\int_{x-L}^{x} f(t)\,dt,$$

where L is a constant representing a fixed interval of time.

This actually involves evaluating this integral over *all* possible intervals $[x - L, x]$ within the interval $[a, b]$. The book has an example where $b - a = 10$ and $L = 3$. That necessitated eight definite integrals to be evaluated. For that reason, it is more appropriate to use software to calculate such values. We will *not* be doing this.