At first glance, $\int 5x^4\left(x^5+3\right)^4\,dx$ looks ghastly. But do you notice anything kinda weird about $5x^4$ and $x^5 + 3$?

Business Calculus
Class notes
Section 4.5: Finding Integrals with Substitution

To find $\int 5x^4\left(x^5+3\right)^4\,dx$ by methods we know, we would need quite a bit of algebra to expand it (and then take the integral term-by-term). We could but it would be tedious and we're liable to make mistakes along the way. There is an easier way. Our new method will be useful for a surprising number of integrals we see in calculus.

The first thing to notice is that $\frac{d}{dx}\left(x^5+3\right)=5x^4$ as was teased above.

Do you remember the Chain Rule?

The second thing we need to know is more subtle. Let's start with a definition of a function $u=x^5+3$. Here, u is said to be a function in x. This means that u is *not* an ordinary variable; it is a *function* with x as its independent variable. Now, what is the derivative of u? Write it as du.

Let's rewrite $\int 5x^4\left(x^5+3\right)^4\,dx$ with this new function u and its derivative in place.

Holy integration, Batman! Did we just create an integral for which we already know its rule? Boom, shakalaka! Find this integral but do *not* forget that we started with x, *not* u.

We are back to indefinite integrals so do *not* forget the constant at the end.

We will practice this procedure and work up to some challenging problems. Another calculus book gives this procedure if you want to take it step-by-step.

PROCEDURE Substitution Rule (Change of Variables)

1. Given an indefinite integral involving a composite function $f(g(x))$, identify an inner function $u = g(x)$ such that a constant multiple of $g'(x)$ appears in the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)\,dx$ in the integral.
3. Evaluate the new indefinite integral with respect to u.
4. Write the result in terms of x using $u = g(x)$.

Disclaimer: Not all integrals yield to the Substitution Rule.

But they do all yield to a good whacking with a solid stick.

expl 1: What, you say you want another?

$$\int 2x\left(x^2+1\right)^4\,dx$$

Along the way, we will need these rules.

$$\int u^p\,du = \frac{u^{p+1}}{p+1} + C,\ p \neq -1$$

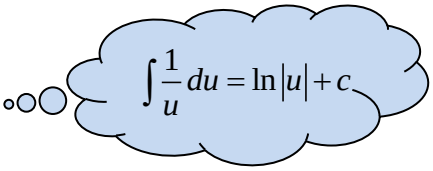
$$\int e^u\,du = e^u + C$$

$$\int \frac{1}{u}\,du = \ln|u| + C$$

If you know that u is nonnegative, disregard the absolute value sign.

expl 2: Evaluate this integral by letting $u = x^4$.

$$\int x^3 e^{x^4}\,dx$$

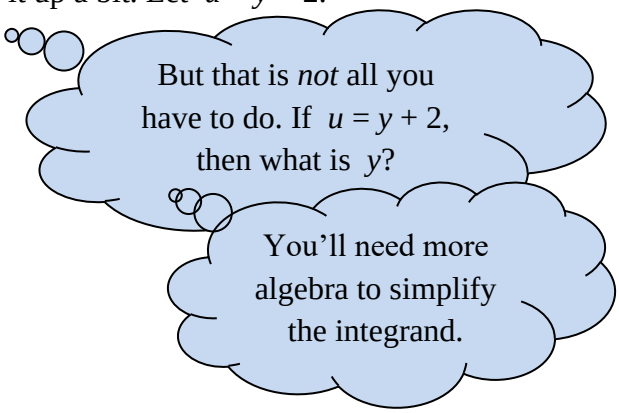


expl 3: Do this one. Let $u = 10x + 3$.

$$\int \frac{1}{10x+3}\,dx$$

expl 4: Here, we use y as our independent variable to mix it up a bit. Let $u = y + 2$.

$$\int y^2 (y+2)^7\,dy$$



expl 5: Use u-substitution to find this integral.

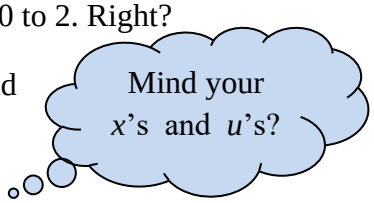
$$\int \frac{(\ln x)^7}{x}\,dx$$

The Return of Definite Integrals:

We started this journey by finding $\int 5x^4\left(x^5+3\right)^4\,dx$. What if we needed $\int_0^2 5x^4\left(x^5+3\right)^4\,dx$?

We have a method for evaluating indefinite integrals. So, we should be able to use our *u*-substitution to find this integral *without* the limits of integration and then use the FUN-damental Theorem of Calculus to finish it out to find the integral from 0 to 2. Right?

There is a small problem. Actually, it can be a *big* problem if you do *not* mind your p's and q's (as my high school algebra teacher loved saying).



It would *not be right* to write $\int_0^2 5x^4\left(x^5+3\right)^4\,dx=\int_0^2 u^4\,du$ using our substitution $u=x^5+3$. Why?

Options for Definite Integrals:

You could just drop the limits of integration from the integrals involving u and then put them back when you get back to x.

Or, referring to the example above, you can find u when x is 0 and when x is 2 and use those as the limits of the integral involving u.

expl 6: Find $\int_0^2 5x^4\left(x^5+3\right)^4\,dx$.