12:00

The slope of a secant line approaches the slope of a tangent line as the two points get closer and closer.

Calculus I
Class notes
The Idea of Limits (section 2.1)

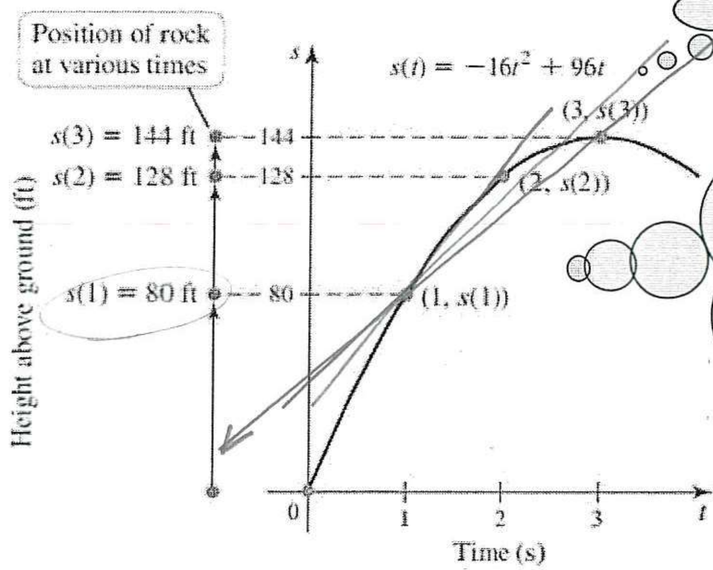
Consider a car that travels 240 miles in 4 hours. We say its **average velocity** is
240 miles / 4 hours = 60 miles per hour.

But that's only an *average*. Sometimes it goes slower and sometimes faster than 60 mph. Its velocity *at any instant* is its **instantaneous velocity**.

Launch a rock upward and let it fall to earth. Let's explore its velocity more closely.

We'll use $s(t) = -16t^2 + 96t$ for the rock's height in feet (also called its **position**) $s(t)$ at time t seconds. Here's a partial graph.

This is *not* the rock's *path*.



Note how *y*-values or heights are determined.

Use a straight edge to draw a **secant line** through the points for $t = 1$ and $t = 3$.

A **secant line** hits graph at two points. A **tangent line** hits only one point.

Find the **average velocity** of the rock by finding the slope of the line through these two points (1, 80) and (3, 144). Include units.

$\frac{\text{rise}}{\text{run}}$

avg vel = slope = $\frac{y_2 - y_1}{x_2 - x_1} = \frac{144 - 80}{3 - 1} = \frac{64}{2} = 32$ feet/sec.

Now, imagine this line tilting. Keep it going through (1, 80) but use, as the second point, another point that is closer. Imagine that second point getting closer and closer to the point (1, 80). You could draw in a couple such lines if you want. What happens to the slope of this line?

Slope of secant lines approach the slope of tangent line.

At some point, our line goes through (1, 80) and another point so close to (1, 80) so that it is indistinguishable from it. We still have two points (of a secant line) and so can find the slope. But notice how it is also very, very close to the slope of the (tangent) line that goes through just the point (1, 80) and no other point on the curve $s(t)$.

Now, the slope of the line that goes through just the point (1, 80) is the **instantaneous velocity** of our rock at time $t = 1$. Again, this line (because it hits only point on the curve) is called the **tangent line**.

We will use the slopes of secant lines to *estimate* the slopes of tangent lines. That is the basic idea of the **derivative**. This concept of looking at the slopes of these lines as the secant lines approach this tangent line introduces us to the idea of a **limit**. Let's put some numbers behind that understanding.

This is table 2.1 from the book. This takes off from where we left it when we found average velocity between $t = 1$ and $t = 3$. Using the language of the table, we found the average velocity for the time interval [1, 3].

expl 1: What number would you say these average velocity values appear to be *approaching*? We will call this the *estimate* of the slope of the tangent line.

64 ft/sec

Table 2.1

Time interval	**Average velocity**
[1, 2]	48 ft/s
[1, 1.5]	56 ft/s
[1, 1.1]	62.4 ft/s
[1, 1.01]	63.84 ft/s
[1, 1.001]	63.984 ft/s
[1, 1.0001]	63.9984 ft/s

Handout: Greek letters:
You will see a lot of Greek letters and this list comes in handy.

Handout: A Manga Guide to Calculus:
This work by H. Kojima and S. Togami reviews the idea of functions and introduces us to differentiation. I will hand out only the introduction of the book. The book (and the whole series) is quite engaging.

2:00 2.1

Introduction to Derivatives:

So, we see how the slope of a secant line will estimate the slope of a tangent line. Let's generalize the process a bit.

$(0,0)$ $(h, -16h^2+96h)$

expl 2: Find the slope of the secant line that goes through the points $(0, s(0))$ and $(h, s(h))$. In other words, we will find the average velocity for the time interval $[0, h]$.

Recall, $s(t) = -16t^2 + 96t$.

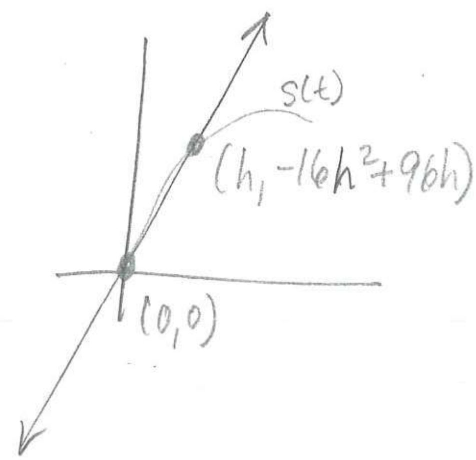
We start with the familiar formula for slope.

$$avg.vel. = slope = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-16h^2+96h-0}{h-0}$$

$$= \frac{-16h^2+96h}{h}$$

$$= \frac{\cancel{h}(-16h+96)}{1\cdot\cancel{h}}$$

$$= -16h+96$$



Notice we did *not* end up at a number like before but rather an expression in h. This calculation gets us pretty far down the road of approximating the slope of the tangent line at the point (0, 0) which is the rock's instantaneous velocity at time $t = 0$.

We will need the idea of a limit which we will develop more fully in the next section. For now, think about finding this slope as h gets closer and closer to the start point of $t = 0$. We will write this as $\lim_{h \to 0}(-16h+96)$.

Pronounced "the limit, as h approaches 0, of -16h + 96".

Let's try another that will circle back toward our first example. We will need to dust off our algebra skills.

$(1, 80)$ $(h, -16h^2+96h)$

expl 3: Find the slope of the secant line that goes through the points $(1, s(1))$ and $(h, s(h))$. In other words, we will find the average velocity for the time interval $[1, h]$.

We start with the familiar formula for slope.

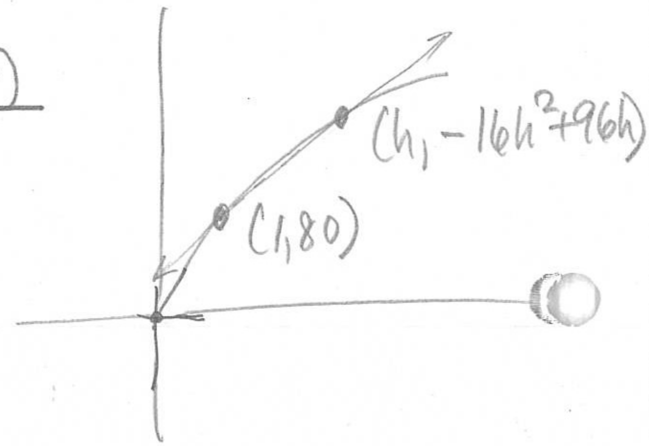
Recall, $s(t) = -16t^2 + 96t$.

$$avg.vel. = slope = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-16h^2 + 96h - 80}{h-1}$$

$$= \frac{-16(h^2 - 6h + 5)}{h-1}$$

$$= \frac{-16(h-5)\cancel{(h-1)}}{1 \cdot \cancel{(h-1)}}$$

$$\text{avg vel.} = -16(h-5)$$



To find the instantaneous velocity at $t = 1$, we would take the limit of this answer as h approaches 1.

Future sections will show us rules to interpret this and many other limits.