2:00

We need a solid, mathematically defined concept for limit. We may see an even more rigorous definition later.

Calculus I
Class notes
Definitions of Limits (section 2.2)

(2.7)

A later section (which we may cover) will add rigor to this concept. For now, this will do. We will define the limit of a function at a certain value of x (independent variable).

Definition: Limit of a Function:
Suppose a function f is defined for all values of x near $x = a$ except possibly at a.

That sounds a little crazy but what we want to picture is a function like this one here. Notice it is defined (shown by the points of the graph) for all values except $x = 3$. So, think of a as any input, even 3.

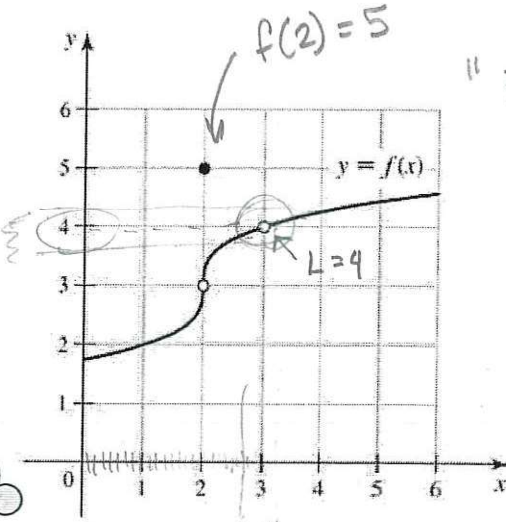


Figure 2.7

" 3 is not in the domain of f "

If $f(x)$ is arbitrarily close to L (meaning as close to L as we would like) for all x sufficiently close (but *not* equal) to a, we write $\boxed{\lim_{x \to a} f(x) = L}$.

Pronounced "the limit, as x approaches a, of $f(x)$".

This will make more sense as you practice.

expl 1: Consider the graph given above. Find the following.

a.) $f(1)$ and $\lim_{x \to 1} f(x)$

$f(1) = 2 \qquad \lim_{x \to 1} f(x) = 2$

b.) $f(2)$ and $\lim_{x \to 2} f(x)$

$f(2) = 5 \qquad \lim_{x \to 2} f(x) = 3$

c.) $f(3)$ and $\lim_{x \to 3} f(x)$

For a limit to exist, the function should approach a single $f(x)$ value from both the left and right sides. We formalize this later.

$f(3)$ dne (does not exist)

$\lim_{x \to 3} f(x) = 4$

$a^2-b^2=(a+b)(a-b)$
Difference of Two Squares

$\frac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}}$

$f(x)=x+2$ except when $x=2$.

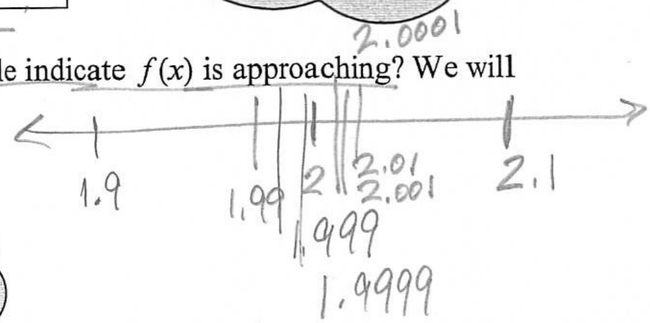
Finding Limits from a Table of Values:

expl 2: Calculate the following values for $f(x)=\frac{x^2-4}{x-2}$ and use it to conjecture about the value of $\lim_{x\to 2} f(x)$.

A little algebra will help us out. Factor the top and simplify.

x	1.9	1.99	1.999	1.9999
$f(x)$	3.9	3.99	3.999	3.9999

As x approaches 2 from the left, what does the table indicate $f(x)$ is approaching? We will write this as $\lim_{x\to 2^-} f(x)$. Practice the notation here.



Do you see the itty bitty "negative sign" to the right of the 2?

$\lim_{x\to 2^-} f(x) = 4$

x	2.1	2.01	2.001	2.0001
$f(x)$	4.1	4.01	4.001	4.0001

As x approaches 2 from the right, what does the table indicate $f(x)$ is approaching? We will write this as $\lim_{x\to 2^+} f(x)$. Practice the notation here.

Do you see the itty bitty "positive sign" to the right of the 2?

$\lim_{x\to 2^+} f(x) = 4$

As we will see in the next theorem, if these two (one-sided) limits agree, then the limit of $f(x)$ as x approaches 2 is that same number. Practice the notation by writing that now.

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = 4$

$\to \lim_{x\to 2} f(x) = 4.$

Let's fill in some details with formal definitions.

12:00

One-sided Limits:

Definition: Right-sided Limit:

Suppose a function f is defined for all values of x near $x = a$ with $x > a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x > a$, we write $\lim_{x \to a^+} f(x) = L$.

Pronounced "the limit, as x approaches a from the right, of $f(x)$ is L"

Definition: Left-sided Limit:

Suppose a function f is defined for all values of x near $x = a$ with $x < a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x < a$, we write $\lim_{x \to a^-} f(x) = L$.

Pronounced "the limit, as x approaches a from the left, of $f(x)$ is L"

As you might have guessed from the last example, we can deduce the limit of $f(x)$ as x approaches a if we know both of these one-sided limits. Here's a theorem.

2:00

Theorem 2.1: Relationship between one-sided and two-sided limits:

Suppose a function f is defined for all values of x near $x = a$ except possibly at a. Then $\lim_{x \to a} f(x) = L$ *if and only if* $\lim_{x \to a^+} f(x) = L$ *and* $\lim_{x \to a^-} f(x) = L$.

p q

The Language of Logic:

"If p, then q" means if p is true, then you know q is also true. The phrase "p *if and only if* q" means both "If p, then q" and, at the same time, "If q, then p". This is equivalent to saying that p and q are the same statement.

So, to find a limit such as $\lim_{x \to 2} f(x)$, we must find *both* $\lim_{x \to 2^+} f(x)$ and $\lim_{x \to 2^-} f(x)$. If they agree, then the limit "as x approaches 2" exists and is this common number.

expl 3: Find the following using the graph to the right.

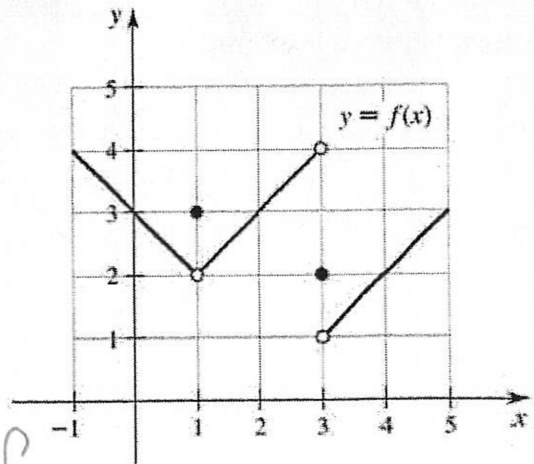
a.) $f(3)$ $= 2$

b.) $\lim_{x \to 3^-} f(x) = 4$

c.) $\lim_{x \to 3^+} f(x) = 1$

d.) $\lim_{x \to 3} f(x)$ = DNE

because $\lim_{x \to 3^-} f \neq \lim_{x \to 3^+} f$.



Graphs and tables can be deceiving or hard to interpret. Be you careful!

expl 4: Create a table of values for $y = \tan\left(3/x\right)$ as started below. Describe the pattern. What happens to $y = \tan\left(3/x\right)$ as x approaches 0? In other words, what is $\lim_{x \to 0} \tan\left(3/x\right)$?

x	$y = \tan\left(3/x\right)$
$12/\pi$	1
$12/3\pi$	-1
$12/5\pi$	1
$12/7\pi$	-1
$12/9\pi$	1

Radian mode

Use the **Table** function on the calculator. Set **Indpt** to **Ask.**

calculator
$12/3\pi \Rightarrow (12/3)*\pi$
Not what we want.

What the heck is going on here? It does *not* look as though the function is approaching any one value as x approaches 0?!? Does graphing the function help?

Try the **Standard** window. Then **Zoom In** more and more. Does that help?

Not one bit!

Eek! It turns out that this limit does *not* exist. It is hard to see that from the graph. We will need other analytical methods to decipher some limits. More on that later.