2:00

> We have explored limits graphically. Now we see how to find them algebraically (analytically).

Calculus I
Class notes
Techniques for Computing Limits (section 2.3)

As we have seen, sometimes the graph of a function cannot be used to find a limit. What's more, a graphical answer is *not* as concrete an answer as the algebraic one. We will be expected to find limits using algebraic means. Several rules will help us along.

Recall: Definition: Continuous function: A function is continuous at a certain x-value if it can be traced from the left of the value to the right of the value without lifting the pencil.

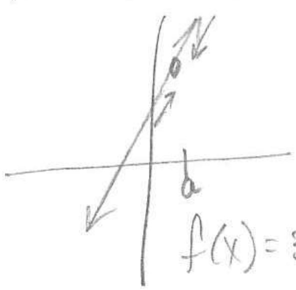
We will explore various types of functions in calculus to find their limits, derivatives, and integrals. We will start with some that you are familiar with, polynomial functions.

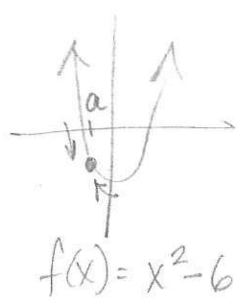
Limits for Polynomial Functions (Linear, Quadratic, Cubic, etc.):

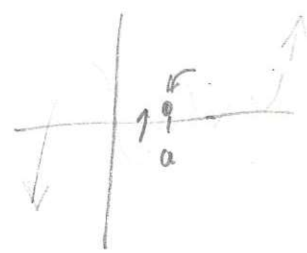
Let $f(x)$ be any polynomial function like the linear $f(x) = mx + b$ or the quadratic $f(x) = ax^2 + bx + c$. (Here, we assume real number coefficients. Recall, a polynomial function is one where the exponents on the x's are all whole numbers.)

Whole numbers $\{0, 1, 2, 3, \ldots\}$

Draw a few examples of polynomial functions here.







Notice how they are continuous across their entire domains. **We have the following truth.**

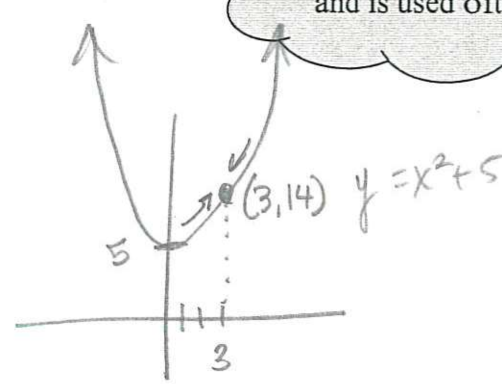
For any polynomial function $f(x)$, we know that $\lim_{x \to a} f(x) = f(a)$ for any x-value $x = a$.

> This is called **direct substitution** and is used often.

expl 1: Find $\lim_{x \to 3} \left(x^2 + 5\right)$.

$f(x) = x^2 + 5$ is a poly fnc

$$\lim_{x \to 3} (x^2 + 5) = 3^2 + 5 = 14$$



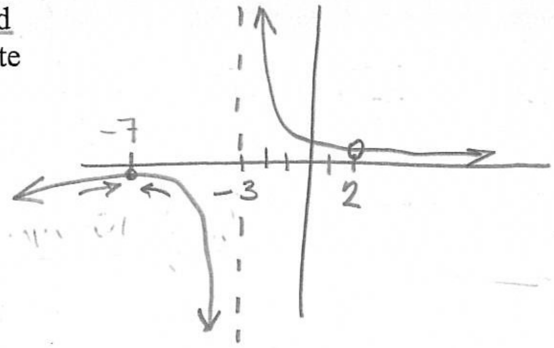
If $q(x)=0$, then the rational fnc is undefined

Limits for Rational Functions (Fraction with Polynomials on Top and Bottom):

Let $p(x)/q(x)$ be a rational function. (Here, $p(x)$ and $q(x)$ are known to be polynomials.)

Recall, this rational function is defined for all values of x except when $q(x)$ is zero.

Recall the shape of a rational function and draw one here. Give it a vertical asymptote at $x = -3$ and a hole at $x = 2$. (There are many correct graphs.)



Can you come up with a likely formula?

$$y = \frac{1\cdot(x-2)}{(x+3)(x-2)}$$

As we see, there may be points of discontinuity on a rational function's graph. **We have this truth.** We know that $\lim_{x\to a}\left(p(x)/q(x)\right) = p(a)/q(a)$ except where $q(a)$ is zero.

There's **direct substitution** again.

We will be working with limits a lot as we travel through calculus and beyond. So, we need a whole slew of rules that hopefully make good sense. Here are a few.

2:00 8/28

Limit Rules:

Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist for two functions f and g. Let c be a real number and n be a positive integer.

We could write $c \in \mathbb{R}, n \in \mathbb{Z}^+$.

$\mathbb{R}$ $\quad$ $\mathbb{Z}$: integers

Sum/Difference: $\lim_{x\to a}\left(f(x) \pm g(x)\right) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

Constant Multiple: $\lim_{x\to a}\left(c\cdot f(x)\right) = c\cdot\lim_{x\to a} f(x)$

Product: $\lim_{x\to a}\left(f(x)\cdot g(x)\right) = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$

Quotient: $\lim_{x\to a}\left(f(x)/g(x)\right) = \lim_{x\to a} f(x) \Big/ \lim_{x\to a} g(x)$ provided $\lim_{x\to a} g(x) \neq 0$

If $n = 2$, square root.
If $n = 3$, cube root.
If $n = 4$, fourth root.
...

Power: $\lim_{x\to a}\left(f(x)\right)^n = \left(\lim_{x\to a} f(x)\right)^n$

Root: $\lim_{x\to a}\left(f(x)\right)^{1/n} = \left(\lim_{x\to a} f(x)\right)^{1/n}$ provided $f(x) > 0$ for x near a, if n is even.

If n is even, then $f(x) > 0$ for x near a in order for $\lim_{x\to a}(f(x))^{1/n} = (\lim_{x\to a} f(x))^{1/n}$ to be true.

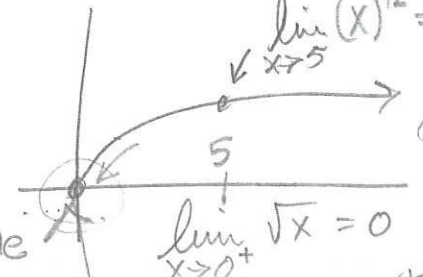
$x^{1/n} = \sqrt[n]{x}$ "nth root of x"

$\lim_{x\to a}(f(x))^{1/n} = \left(\lim_{x\to a} f(x)\right)^{1/n}$

A Closer Look at the Limit of a Root Rule:

Let's examine that one for roots a little closer. Consider $f(x)=x$. If $n=2$, we are dealing with the square root of x or $x^{1/2}=\sqrt{x}$. When we explore its graph to, say, find $\lim_{x\to 0}(x)^{1/2}=\lim_{x\to 0}\sqrt{x}$, what happens? (Do *not* try to use the rule yet. Just graph.)

Can you draw the graph of $y=\sqrt{x}$ from memory? What is this limit? Do you recall how we said we find limits in the previous section?



$\lim_{x\to0}\sqrt{x}$ dne

$\lim_{x\to0^-}\sqrt{x}$ dne

$\lim_{x\to0^+}\sqrt{x}=0$

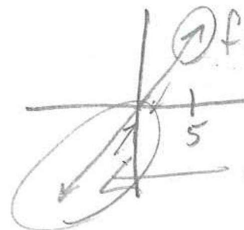
Let's *try* to use the rule now, which says that $\lim_{x\to 0}(x)^{1/2}=\left(\lim_{x\to 0}(x)\right)^{1/2}$ provided $f(x)>0$ for x near 0, since $n=2$ is even. (Our value of a is 0.) But wait!

this is not true.

when $x<0$, $f(x)\not>0$.

Since $f(x)=x$ is *not* positive to the left of 0, the rule does *not* apply for this example since n is even.

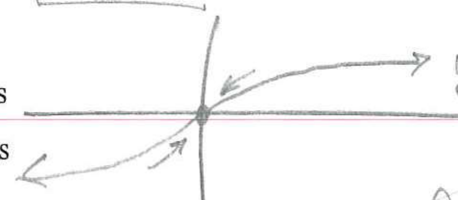
Draw the graph of $y=x$.



$f(x)\not>0$ for all x near $a=0$.

We see this issue does *not* arise when n is odd. Consider again the same $f(x)=x$. If we take $n=3$, we are dealing with the cube root of x or $x^{1/3}=\sqrt[3]{x}$.

When we explore its graph to find $\lim_{x\to 0}(x)^{1/3}=\lim_{x\to 0}\sqrt[3]{x}$, what happens? Draw this graph from memory. What is the value of this limit?



$\lim_{x\to0^-}\sqrt[3]{x}=0$ $\quad$ $\lim_{x\to0^+}\sqrt[3]{x}=0$ $\rightarrow$ $\lim_{x\to0}\sqrt[3]{x}=0$

because left and right limits agree

Notice that since $n=3$ is *not* even, we do *not* need to adhere to the criteria that $f(x)>0$ for x near $a=0$. And, indeed, we can find this limit from the graph of $y=x^{1/3}=\sqrt[3]{x}$ or the rule as $\left(\lim_{x\to 0}(x)\right)^{1/3}$. Use the rule now to verify this limit.

Ah, math!

$\lim_{x\to0}\sqrt[3]{x}=\lim_{x\to0}x^{1/3}=\left(\lim_{x\to0}x\right)^{1/3}$

$=0^{1/3}=\sqrt[3]{0}=0$

expl 2: Find the following limits and state the rule(s) used. Show the steps explicitly. Assume that (for some unknown functions) $\lim_{x \to 1} f(x) = 3$ and $\lim_{x \to 1} g(x) = 8$.

a.) Find $\lim_{x \to 1}(5f(x))$. $= 5 \cdot \lim_{x \to 1} f(x) = 5 \cdot 3 = 15$

Constant multiple rule.

b.) Find $\lim_{x \to 1}(f(x) - g(x))$. $= \lim_{x \to 1} f(x) - \lim_{x \to 1} g(x) = 3 - 8 = -5$

Sum/difference rule

c.) Find $\lim_{x \to 1}(g(x))^{2/3}$.

$= \left(\lim_{x \to 1} g(x)\right)^{2/3} = 8^{2/3} = \left(\sqrt[3]{8}\right)^2 = 4$

Power and Root Rules

Using the Polynomial and Rational Rules:

expl 3: Find the following limits.

There's **direct substitution** again.

a.) $\lim_{x \to 3}\left(2x^2 + 4x^3 - 7\right) = 2 \cdot 3^2 + 4 \cdot 3^3 - 7$

$= 119$

b.) $\lim_{x \to 3} \frac{\left(2x^2 + 4x^3 - 7\right)}{x+5} = \frac{2 \cdot 3^2 + 4 \cdot 3^3 - 7}{3+5} = \frac{119}{8}$

Rational fnc Rule (pg 2 top)

OR ≈ 14.9 rounded to the nearest 10th.

12:00

When Direct Substitution Fails:

Notice when we try direct substitution in the next examples, we get an undefined answer. That will not do! In a later section, we will have yet another method to deal with this in more general problems. However, sometimes problems are written in such a way so that we can use a little algebra to get out of that pickle.

Whip out factoring from your bag of tricks to rewrite the function.

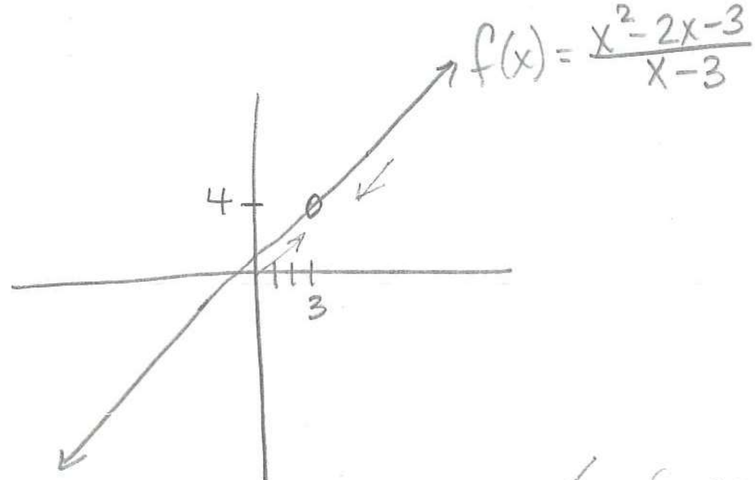
expl 4: Find the following limit.

$$\lim_{x\to 3}\frac{x^2-2x-3}{x-3} = \lim_{x\to 3}\frac{\cancel{(x-3)}(x+1)}{1\cdot\cancel{(x-3)}}$$

$$= \lim_{x\to 3}(x+1)$$

$$= 3+1 = 4$$

Let's look at the graph of $f(x)=\dfrac{x^2-2x-3}{x-3}$. Draw it on your own or use the calculator. (Don't forget its hole.) From the graph, does the limit you got above make sense?



expl 5: Find the following limit.

Here's another to put in your bag of tricks. Rewrite the function by multiplying both top and bottom by $\sqrt{x}+1$.

conjugate of $\sqrt{x}-1$

$$\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1}$$

$$\lim_{x\to 1}\frac{(x-1)(\sqrt{x}+1)}{(\sqrt{x}-1)(\sqrt{x}+1)} \quad \text{FOIL}$$

$$= \lim_{x\to 1}\frac{\cancel{(x-1)}(\sqrt{x}+1)}{\cancel{x-1}}$$

$$= \lim_{x\to 1}(\sqrt{x}+1) = \lim_{x\to 1}\sqrt{x} + \lim_{x\to 1}1$$

$$= \left(\lim_{x\to 1}x\right)^{1/2} + \lim_{x\to 1}1 = 1+1 = 2$$

One-sided Limits:

All of the rules we have had so far (*except* for the one for roots) do work for one-sided limits too. So, if you are asked about $\lim\limits_{x\to a^-} f(x)$ or $\lim\limits_{x\to a^+} f(x)$, know that the rules still work. However, we will amend the Roots Rule.

One-sided Limit for Roots:

Assume $n \in \mathbb{Z}^+$ again. Also, these statements about $f(x)$ only apply if we are dealing with even roots, where n is an even number. If n is odd, we do *not* need to verify that $f(x)$ is non-negative.

a.) **Right-sided Limit for Roots:**

$\lim\limits_{x\to a^+} \left(f(x)\right)^{1/n} = \left(\lim\limits_{x\to a^+} f(x)\right)^{1/n}$ provided $f(x) \ge 0$ for x near a for $x > a$, if n is even.

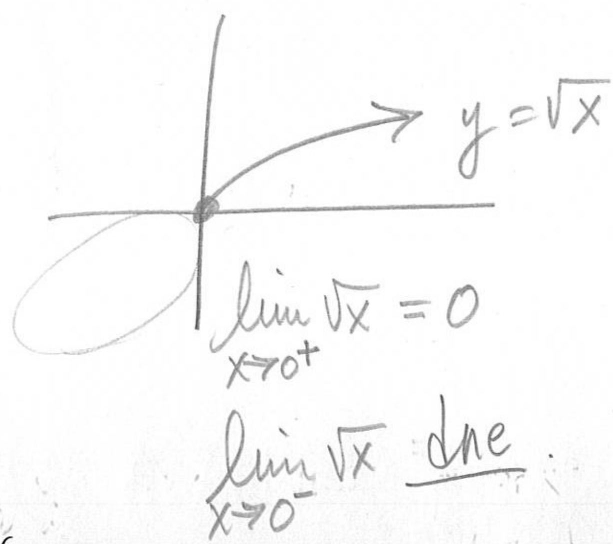
b.) **Left-sided Limit for Roots:**

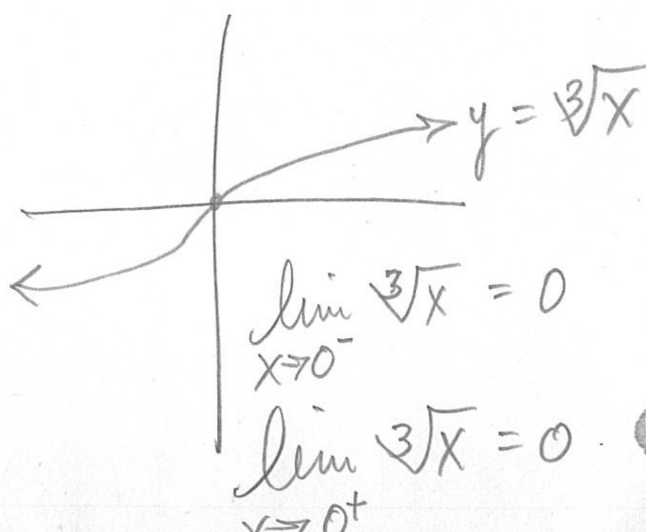
$\lim\limits_{x\to a^-} \left(f(x)\right)^{1/n} = \left(\lim\limits_{x\to a^-} f(x)\right)^{1/n}$ provided $f(x) \ge 0$

for x near a for $x < a$, if n is even.

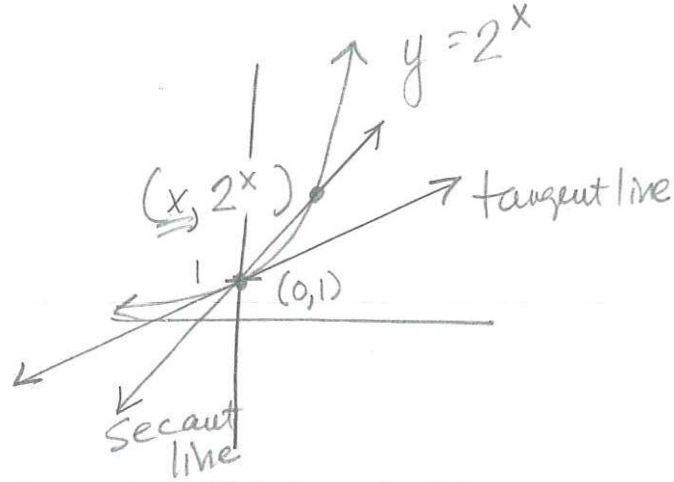
> What changed? Now, $f(x)$ is allowed to be 0 as well. Also, we added that this only matters on the side of a where the limit is being found.

Give yourself a quick drawing of $y = \sqrt{x}$ and you'll see that the right-sided limit exists but the left-sided one does *not*. This is, once again, *not* an issue for $y = \sqrt[3]{x}$ which uses an odd value for n. Grace the page with that graph too please.





when $x = 0$, $y = 2^0 = 1$



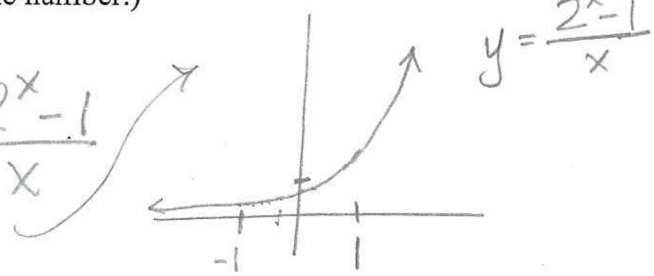
Exponential Functions (No Rules Yet!):

Consider the graph of $y = 2^x$. Give yourself a quick drawing here.

What if we wanted the slope of the tangent line to this curve at $x = 0$?

Plot and label two points on your curve, (0, 1) and a generic $(x, 2^x)$ further to the right. Find the slope of the secant line. (We only need a formula, *not* some number.)

Slope of secant line $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{2^x - 1}{x - 0} = \dfrac{2^x - 1}{x}$



Again, as before, we imagine this second point getting closer and closer to the point (0, 1). As this point moves to the left, the secant line comes closer and closer to coinciding with the tangent line at (0, 1). We use limits to describe this.

We say that the slope of this tangent line is, in fact, $\lim_{x \to 0} \dfrac{2^x - 1}{x}$. Can you find it with direct substitution? We have *no* rules that apply here. What's a student to do?

Let's use good old plug and chug with a helpful little table. Consider the completed tables here.

x	-1	-0.1	-0.01	-0.001	-0.000 1	-0.000 01
$\dfrac{2^x - 1}{x}$	0.5	0.66967	0.69075	0.69291	0.69312	0.69314

$\lim_{x \to 0^-} \dfrac{2^x - 1}{x} \approx 0.693$

The top table explores the left-sided limit and the bottom table explores the right-sided limit. What are both approaching? Round to three decimal places.

x	1	0.1	0.01	0.001	0.000 1	0.000 01
$\dfrac{2^x - 1}{x}$	1.0	0.71773	0.69556	0.69339	0.69317	0.69315

What would you say the slope of the tangent line at (0, 1) is? Write it in limit notation.

$\lim_{x \to 0^+} \dfrac{2^x - 1}{x} \approx 0.693 \rightarrow$ Slope of tangent line $= \lim_{x \to 0} \dfrac{2^x - 1}{x} \approx 0.693$

The Squeeze Theorem:

This one will come in handy in a few places. Take a look at this picture.

Two functions $f(x)$ and $h(x)$ have the same limit as x approaches a.

It so happens that the function $g(x)$ is in between these two other functions in the area around $x = a$.

What can be said about the limit of $g(x)$ as x approaches a? Read below…

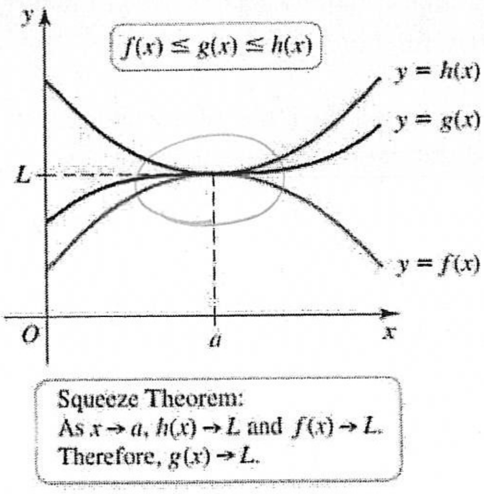


Figure 2.19

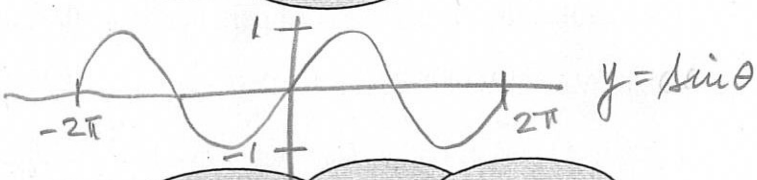
Assume functions f, g, and h satisfy $f(x) \le g(x) \le h(x)$ for all values of x near a, except possibly at a.

If $\lim_{x\to a} f(x) = L$ and also $\lim_{x\to a} h(x) = L$,

then we know that $\lim_{x\to a} g(x) = L$.

That last bit leaves open the possibility that some function is *not* defined at $x = a$.

Trigonometric Limits:



expl 6: Find the following crazy looking limit.

$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$

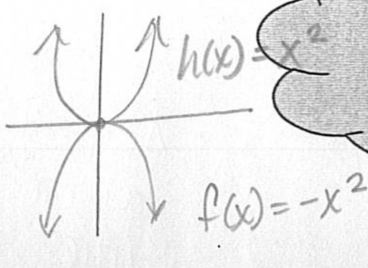
Do you recall the graph of $y = \sin(\theta)$? Would you agree that $-1 \le \sin(\theta) \le 1$ and hence $-1 \le \sin\left(\frac{1}{x}\right) \le 1$ for $x \ne 0$?

This last inequality needs only to be multiplied by x^2 to see that $-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$ for $x \ne 0$.

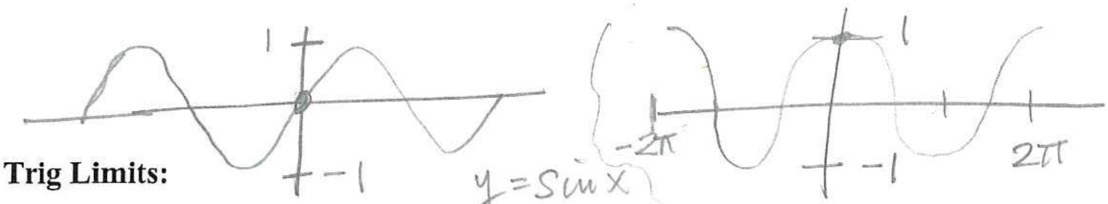
Do you see how the Squeeze Theorem can be applied? Do you recall the graphs of $y = x^2$ and $y = -x^2$?

From the theorem, use $f(x) = -x^2$, $h(x) = x^2$.

So, $\lim_{x\to 0}(-x^2) = 0$ and $\lim_{x\to 0}(x^2) = 0$. Since $f(x) \le g(x) \le h(x)$, we know from the theorem that $\lim_{x\to 0} g(x) = 0$ also.



Hence, $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$



Some Common Trig Limits:

This Squeeze Theorem can be used to prove a couple of important limits we include here. We will then use these facts, along with the limit rules given earlier and other tidbits from our vast memory, to find other limits.

Limits of Sine and Cosine:

$$\lim_{x\to 0}(\sin x) = 0 \qquad \lim_{x\to 0}(\cos x) = 1$$

Picture the graphs of sine and cosine and you'll see why these must be true.

expl 7: Find the following limit.

$$\lim_{x\to 0}\frac{(\sin 2x)}{\sin x} = \lim_{x\to 0}\frac{2\cancel{\sin x}\cos x}{\cancel{\sin x}}$$

$$= \lim_{x\to 0}(2\cdot\cos x)$$

$$= 2\cdot\lim_{x\to 0}(\cos x)$$

$$= 2\cdot 1$$

$$= 2$$

constant multiple rule $c=2$

Trig identity:
$\sin 2\theta = 2\sin\theta\cos\theta$

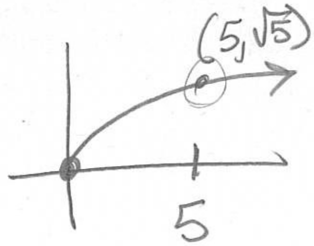
Handout: Trig Cheat Sheet:
This comes from Paul Dawkins (my internet hero) and is quite nice to keep close by as it has all those bits you may have forgotten from Trig.

Handout: Limits:
Again from Paul Dawkins, but this time he has listed all those rules we will need for limits. Some of this sheet will come up later.

Worksheet: Have you reached your limit?:
This is a practice sheet I made up.

Question: Give an example where left and right limits exist and agree.

Addendum to pg 3

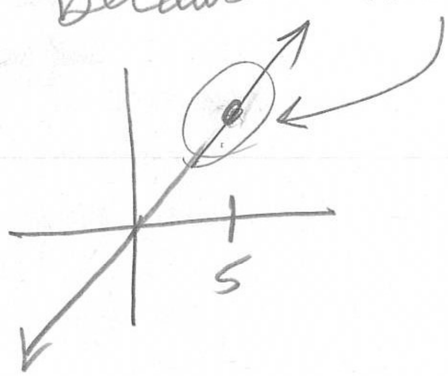


$y = x^{1/2}$

$$\lim_{x \to 5} x^{1/2} = \sqrt{5}$$

from our graph of $y = \sqrt{x}$

OR using the rule which applies because $f(x) = x > 0$ for x near 5.



Our rule would say

$$\lim_{x \to 5} x^{1/2} = \left(\lim_{x \to 5} x\right)^{1/2}$$

$$= 5^{1/2}$$

$$= \sqrt{5}$$

which matches