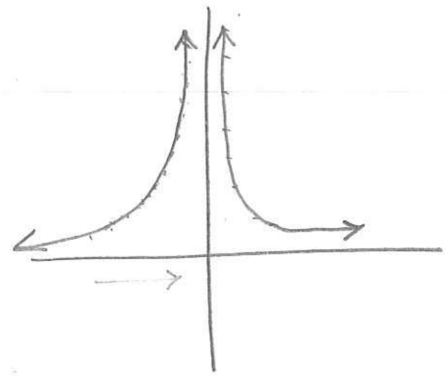
12:00

How do we show if a graph soars off toward infinity as x approaches some number like 0? How would that look?

Calculus I
Class notes
Infinite Limits (section 2.4)

To infinity and beyond! Consider the function $y = {}^{1}\!/_{x^2}$. Graph it on the Standard Window and copy it here.



As we approach 0 from the left, what does the function (y-values) go toward? ∞

As we approach 0 from the right, what does the function (y-values) go toward?

∞

We are talking about $\lim\limits_{x\to 0^-} {}^{1}\!/_{x^2}$ and $\lim\limits_{x\to 0^+} {}^{1}\!/_{x^2}$.

Again, recall that we define $\lim\limits_{x\to 0} {}^{1}\!/_{x^2}$ from the values of $\lim\limits_{x\to 0^-} {}^{1}\!/_{x^2}$ and $\lim\limits_{x\to 0^+} {}^{1}\!/_{x^2}$. What would you say is the value of $\lim\limits_{x\to 0} {}^{1}\!/_{x^2}$? $\lim\limits_{x\to 0} \frac{1}{x^2} = \infty$

Lovely. Just lovely. Now, we will certainly write this as being "equal to infinity". However, let's be clear. A limit is, strictly speaking, a number and so is *not* really equal to infinity. This limit *does not, in fact, exist* but we will write such limits as infinity or negative infinity.

Soon, we will also consider stuff like $\lim\limits_{x\to \infty} {}^{1}\!/_{x^2}$. — section 2.5
We'll need to keep those ideas separate.

Definition: Infinite Limits:

Suppose f is defined for all x near a. If $f(x)$ grows arbitrarily large for all x sufficiently close (but *not* equal) to a, we write $\lim\limits_{x\to a} f(x) = \infty$. We pronounce this as "the limit of f, as x approaches a, is infinity".

Recall that both $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ have to agree.

If $f(x)$ is negative and grows arbitrarily large in magnitude for all x sufficiently close (but *not* equal) to a, we write $\lim\limits_{x\to a} f(x) = -\infty$.

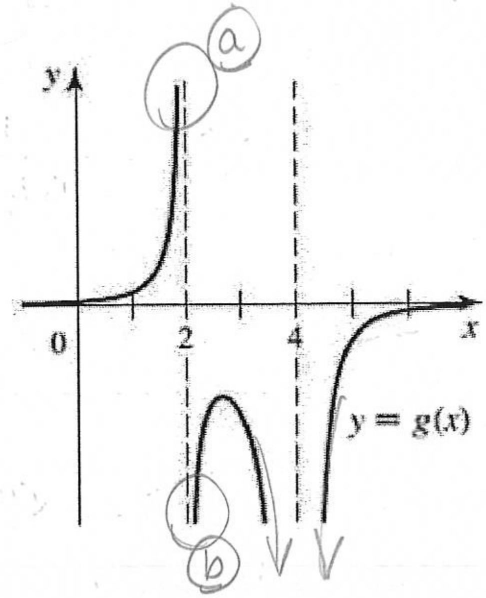
We pronounce this as "the limit of f, as x approaches a, is negative infinity".

The Limit is Infinity but Does *Not* Exist?

Yes, technically the limit does *not* exist but this gives us a handy notation to denote a graph that soars up or down towards infinity or its negative. This limit is said to *not* exist. We will still write this limit as equal to infinity or its negative if $f(x)$ approaches it as we approach a from the right *and* the left.

If the right-sided limit does *not* match the left-sided limit, we will say the limit does *not* exist at all. We often use "d.n.e." as an abbreviation.

expl 1: Notice the vertical asymptotes in this function's graph. Find the following limits. Some are left-sided, some are right-sided, and some are neither. Say "dne" if the limit does *not* exist; use ∞ or -∞ when appropriate.



a.) $\lim_{x\to 2^-} g(x) = \infty$

b.) $\lim_{x\to 2^+} g(x) = -\infty$

c.) $\lim_{x\to 2} g(x)$ dne

d.) $\lim_{x\to 4^-} g(x) = -\infty$

e.) $\lim_{x\to 4^+} g(x) = -\infty$

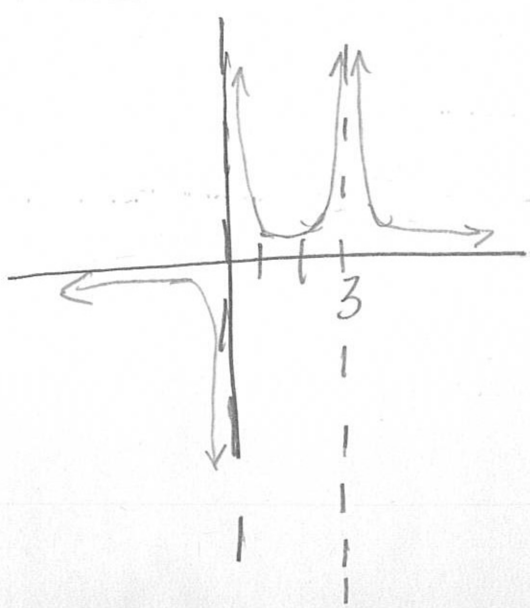
f.) $\lim_{x\to 4} g(x) = -\infty$

~~Recall~~: Definition: Vertical Asymptotes:

You will be familiar with these through rational functions. We define it here using our new concept of limit.

If $\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a^-} f(x) = \pm\infty$, or $\lim_{x\to a^+} f(x) = \pm\infty$, then the line $x = a$ is a vertical asymptote of $f(x)$.

expl 2: Draw a graph of a function such that $\lim_{x\to 3} f(x) = \infty$ and $\lim_{x\to 0^+} f(x) = \infty$ but $\lim_{x\to 0^-} f(x) = -\infty$.



Finding Limits Algebraically (Analytically):

expl 3: Find these limits without consulting a graph.

Picture $z = 3.1$

a.) $\lim_{z\to 3^+} \frac{(z-1)(z-2)}{(z-3)}$ → $\frac{pos \cdot pos}{pos}$

$\lim_{z\to 3^+} \frac{(z-1)(z-2)}{(z-3)} = +\infty$

b.) $\lim_{z\to 3^-} \frac{(z-1)(z-2)}{(z-3)}$ → $\frac{pos * pos}{neg}$

Picture $z = 2.9$

$\lim_{z\to 3^-} \frac{(z-1)(z-2)}{(z-3)} = -\infty$

Direct substitution gets us 0 on bottom, no joy.

Without using a specific value, imagine a z-value that is slightly more than 3.

For each factor, would it be positive or negative?

Part b, do the same for a number that you imagine slightly less than 3…

c.) $\lim_{z\to 3} \frac{(z-1)(z-2)}{(z-3)}$ dne because

$\lim_{z\to 3^+} \frac{(z-1)(z-2)}{(z-3)} \neq \lim_{z\to 3^-} \frac{(z-1)(z-2)}{(z-3)}$

Division and Infinity:

When we have a fraction whose top is *way bigger* than its bottom, the quotient is a big number. See this with a fraction like $5/0.03$. As the bottom gets smaller and smaller, the quotient gets bigger and bigger. Look at $5/0.003$, $5/0.0003$, $5/0.000\,03$, … $\to +\infty$

If we let the bottom approach 0, the fraction will approach positive (or negative) infinity. You want to be on the lookout for fractions whose bottom approaches 0 while its top does *not*.

direct substitution doesn't work.

expl 4: Find these limits but you'll need your factoring skills.

a.) $\lim_{x\to 0}\dfrac{x-3}{x^4-9x^2} = \lim_{x\to 0}\dfrac{1(x-3)}{x^2(x^2-9)} = \lim_{x\to 0}\dfrac{1\cancel{(x-3)}}{x^2\cancel{(x-3)}(x+3)} = \lim_{x\to 0}\dfrac{1}{x^2(x+3)}$

So, we'll need $\lim_{x\to 0^-}\dfrac{1}{x^2(x+3)} \rightarrow \dfrac{1}{(pos)(pos)} \rightarrow \lim_{x\to 0^-}\dfrac{1}{x^2(x+3)} = +\infty$

Picture $x=-0.1$

and $\lim_{x\to 0^+}\dfrac{1}{x^2(x+3)} \rightarrow \dfrac{1}{(pos)(pos)} \rightarrow \lim_{x\to 0^+}\dfrac{1}{x^2(x+3)} = +\infty$

Picture $x=0.1$

$$\boxed{\lim_{x\to 0}\dfrac{1}{x^2(x+3)} = +\infty}$$

$$\lim_{x\to 0}\dfrac{x-3}{x^4-9x^2} = +\infty$$

b.) $\lim_{x\to 3}\dfrac{x-3}{x^4-9x^2} = \lim_{x\to 3}\dfrac{1}{x^2(x+3)}$

$= \dfrac{1}{3^2(3+3)} = \dfrac{1}{9\cdot 6} = \dfrac{1}{54}$

c.) $\lim_{x\to -3}\dfrac{x-3}{x^4-9x^2} = \lim_{x\to -3}\dfrac{1}{x^2(x+3)}$

Part *c*, you'll need to find left and right-sided limits first.

So, we need $\lim_{x\to -3^-}\dfrac{1}{x^2(x+3)} \rightarrow \dfrac{1}{(pos)(neg)} \rightarrow \lim_{x\to -3^-}\dfrac{1}{x^2(x+3)} = -\infty$

picture $x=-3.1$

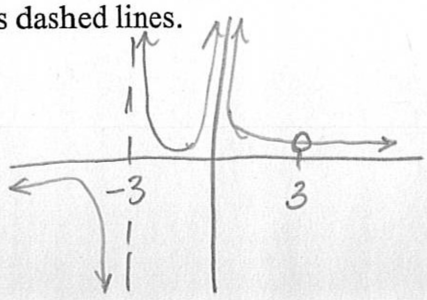
and $\lim_{x\to -3^+}\dfrac{1}{x^2(x+3)} \rightarrow \dfrac{1}{(pos)(pos)} \rightarrow \lim_{x\to -3^+}\dfrac{1}{x^2(x+3)} = +\infty$

picture $x=-2.9$

$$\lim_{x\to -3}\dfrac{x-3}{x^4-9x^2} \text{ dne}$$

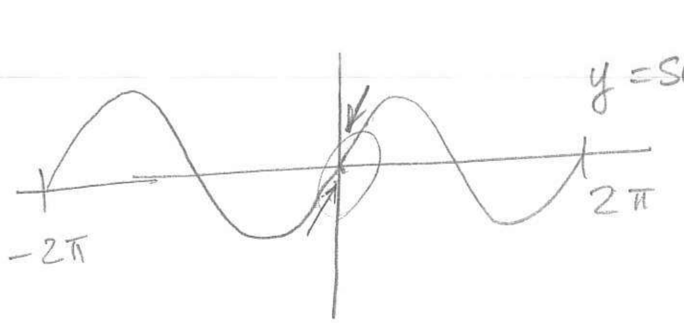
Check by graphing the function $y=\dfrac{x-3}{x^4-9x^2}$.

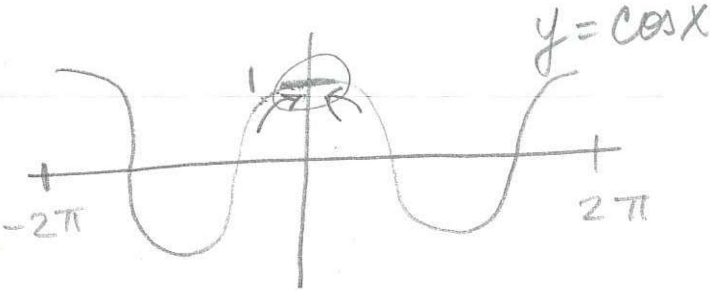
Copy vertical asymptotes as dashed lines.
Label the hole at $x=3$.



Trig Limits:

Once again, it helps to have a clear picture of the sine and cosine curves. Draw them now.





expl 5: Find $\lim_{\theta\to 0}\left(\frac{2+\sin\theta}{1-\cos^2\theta}\right)$. $\Rightarrow$ direct substitution $\rightarrow \frac{2}{0}$

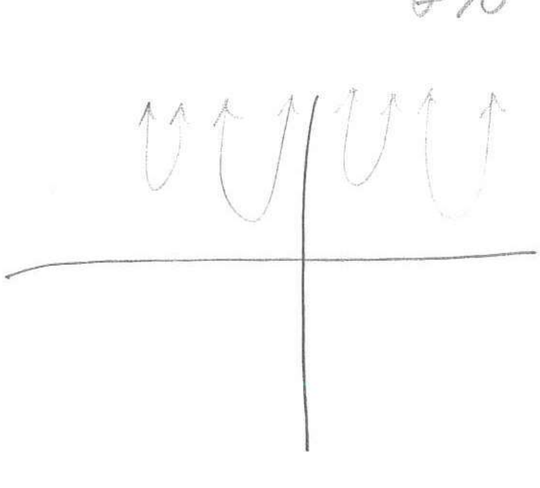
So, we need $\lim_{\theta\to 0^-}\left(\frac{2+\sin\theta}{1-\cos^2\theta}\right) \Rightarrow \frac{\text{approaches 2}}{\text{approaches 0 but is pos.}} \rightarrow +\infty$

Picture $\theta = -0.1$

and $\lim_{\theta\to 0^+} \frac{2+\sin\theta}{1-\cos^2\theta} \rightarrow \frac{\text{approaching 2}}{\text{approaches 0 but is pos}} \rightarrow +\infty$

picture $\theta = 0.1$

So, $\lim_{\theta\to 0} \frac{2+\sin\theta}{1-\cos^2\theta} = +\infty$



$[-2\pi, 2\pi] \times [-10, 10]$