2:00

What do y-values approach as x approaches infinity? In other words, what is happening at the ends of the graph?

Calculus I
Class notes
Limits at Infinity (section 2.5)

In the last section, we explored limits such as $\lim_{x\to 0} \frac{1}{x^2}$ and noticed that the answers could be infinity or its negative. Interesting. As x approached some finite number, the y-values approached infinity or its negative. Let's switch the question around a bit and ask, "What happens to the y-values as x approaches infinity?"

We have actually answered this question before in algebra class. You have studied the end behavior of polynomial functions and horizontal or oblique asymptotes of rational functions. So, we will play with those kinds of functions but also explore this question with regard to trigonometric or transcendental functions.

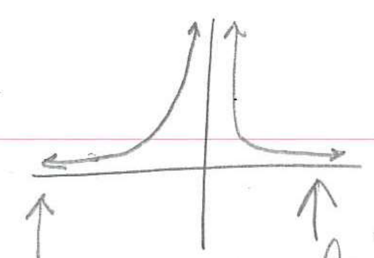
Definition: Transcendental function: A function that *cannot* be written as a finite combination of addition, subtraction, multiplication, division, raising to a power, or roots. Examples include $y = e^x$, $y = \log_e x$, or trig functions.

We will explore these at the end of the Notes. First, let's deal with some more familiar functions.

A fun application of the question we answer here might be to figure out if a population or oscillating structure reaches a steady state as time progresses.

expl 1: Use a graph to determine $\lim_{x\to\infty} 2x^{-8}$ and $\lim_{x\to -\infty} 2x^{-8}$.

Start on the Standard Window but you'll want to change your window to something like [-10, 10] x [0, 0.001].



$\lim_{x\to -\infty} 2x^{-8} = 0$ $\lim_{x\to\infty} 2x^{-8} = 0$

Recall this function can also be written as $y = \frac{2}{x^8}$. Do you remember the horizontal asymptotes of rational functions?

Definition: Limits at Infinity and Horizontal Asymptotes:

If $f(x)$ becomes arbitrarily close to a finite number L for all sufficiently large and positive x, then we write $\lim_{x\to\infty} f(x) = L$. The line $y = L$ is a **horizontal asymptote** of $f(x)$.

The same thing goes for $\lim_{x\to-\infty} f(x) = M$ where we would say $f(x)$ becomes arbitrarily close to a finite number M for all *negative* x-values that are "sufficiently large in magnitude".

We use M here to indicate that the two limits are *not* necessarily the same number (but will be in the case of rational functions).

Here is an example of a function where the two limits are different.

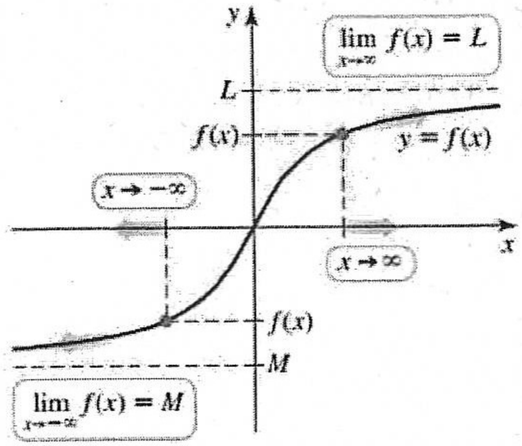


Figure 2.30

We will use the rules we were given in an earlier section. Go term-by-term.

L2.3

$$\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

Algebraically (Analytically) Finding Limits:

expl 2: Find the following limit.

$$\lim_{x\to\infty}\left(5+\frac{1}{x}+\frac{10}{x^2}\right)$$

$$= \lim_{x\to\infty}(5) + \lim_{x\to\infty}\left(\frac{1}{x}\right) + \lim_{x\to\infty}\left(\frac{10}{x^2}\right)$$

$$= 5 + 0 + 0$$

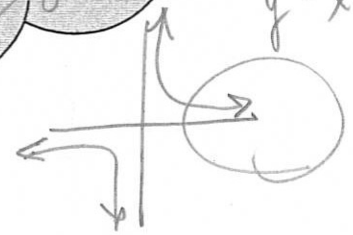
$$= 5$$

When thinking about $y = 1/x$, we imagine x getting bigger and bigger. Describe what is happening to the values

$1/10, 1/100, 1/1000, 1/10000, \ldots$

0

$y = \frac{1}{x}$



from 2.3 $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} f(x) \Big/ \lim_{x \to a} g(x)$

Trig Limits:

expl 3: Find the following limit.

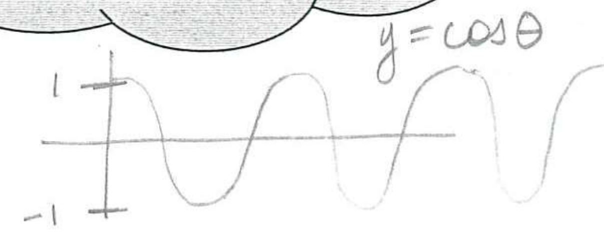
$$\lim_{\theta \to \infty} \frac{\cos\theta}{\theta^2}$$

$$= \frac{\lim_{\theta \to \infty} \cos\theta}{\lim_{\theta \to \infty} \theta^2}$$

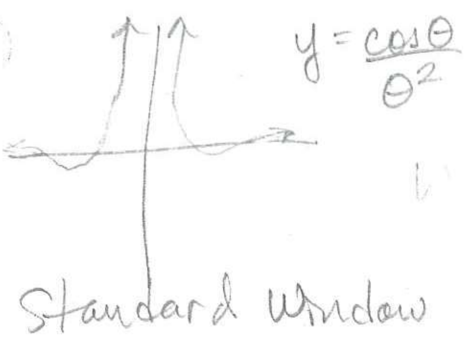
$$= \frac{\text{oscillates from } -1 \text{ to } 1}{\text{approaches } \infty} = 0$$

Give yourself a quick graph of $y = cos(\theta)$. Use a limit rule to rewrite this as the quotient of limits. For the bottom, what would θ^2 approach as θ approached ∞?

If the top of a fraction is somewhere between -1 and 1 but the bottom approaches ∞, what does the fraction approach?



For an added bonus of "Wha?", try graphing the function $y = \frac{\cos x}{x^2}$ first on the Standard Window and then on the window [-50, 50] x [-0.01, 0.01]. Be sure you are in Radian MODE.



Is the universe playing tricks on us?

Infinite Limits at Infinity:

Wha??

What functions have you seen that satisfy $f(x)$ approaches positive or negative infinity as x does the same?

Definition: Infinite Limits at Infinity: If $f(x)$ becomes arbitrarily large as x becomes arbitrarily large, then we write $\lim_{x \to \infty} f(x) = \infty$. Similar language will be used to describe

$\lim_{x \to -\infty} f(x) = \infty$, $\lim_{x \to \infty} f(x) = -\infty$, and $\lim_{x \to -\infty} f(x) = -\infty$.

Again, these limits *do not exist* but this is a handy notation.

Do you recall the four types of end behavior for polynomial functions?

Let $f(x) = 6x^8 + 9x^{10} + 4x - 6x^2 + 13$

$9x^{10}$ is leading term

9 is lead coeff., 10 is the degree of the poly

Recall: End Behavior of Polynomial Functions:

The end behavior (how the y-values behave at the ends of the graph) of polynomial functions is solely determined by the leading term (term with the highest exponent). The **leading coefficient** is the coefficient of this term. The **degree** is this highest exponent.

Below is the **Leading Term Test,** used to determine the end behavior of any polynomial function. If you need more review, ask me and I will direct you to my Algebra Worksheet **Polynomial Functions: End Behavior** on www.stlmath.com.

	Leading coefficient is negative	**Leading coefficient is positive**
Degree is odd	as $x \to -\infty$, $y \to \infty$ as $x \to \infty$, $y \to -\infty$	as $x \to -\infty$, $y \to -\infty$ as $x \to \infty$, $y \to \infty$
Degree is even	as $x \to -\infty$, $y \to -\infty$ as $x \to \infty$, $y \to -\infty$	as $x \to -\infty$, $y \to \infty$ as $x \to \infty$, $y \to \infty$

$y = x^3$

$y = x^2$

Here, we write this knowledge in terms of limits at infinity.

THEOREM 2.6 Limits at Infinity of Powers and Polynomials

Let n be a positive integer and let p be the polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$, where $a_n \neq 0$.

n = degree of poly

1. $\lim_{x \to \pm\infty} x^n = \infty$ when n is even.
2. $\lim_{x \to \infty} x^n = \infty$ and $\lim_{x \to -\infty} x^n = -\infty$ when n is odd.
3. $\lim_{x \to \pm\infty} \frac{1}{x^n} = \lim_{x \to \pm\infty} x^{-n} = 0$.
4. $\lim_{x \to \pm\infty} p(x) = \lim_{x \to \pm\infty} a_n x^n = \pm\infty$, depending on the degree of the polynomial and the sign of the leading coefficient a_n.

Recall the way a polynomial is written. The term $a_n x^n$ is the leading term where a_n is the leading coefficient and n is the degree.

expl 4: Find the following limits.

degree = 7 (odd)
lead coeff = 3 (positive)

Only the leading terms need be considered.

a.) $\lim_{x\to-\infty}\left(3x^7+x^2\right) = \lim_{x\to-\infty}\left(3x^7\right) = -\infty$

b.) $\lim_{x\to\infty}\left(3x^{12}-9x^4\right) = \lim_{x\to\infty}\left(3x^{12}\right) = \infty$

Limits at Infinity for Rational Functions:

After some practice, you will be able to do some of these intuitively. Remembering the rules for horizontal and oblique asymptotes for rational functions will certainly help.

A good technique is to divide *all* of the terms in the rational function by x^n where n is the degree of the denominator. You then deal with the limit term-by-term. Let's try that before we explore the rules.

expl 5a: Find the following limit.

Divide every term by x^4.

$\lim_{x\to\infty}\frac{2x+1}{3x^4-2}$

$$= \lim_{x\to\infty}\frac{\frac{2x}{x^4}+\frac{1}{x^4}}{\frac{3x^4}{x^4}-\frac{2}{x^4}} = \lim_{x\to\infty}\frac{\frac{2}{x^3}+\frac{1}{x^4}}{3-\frac{2}{x^4}} = \frac{\lim_{x\to\infty}\frac{2}{x^3}+\lim_{x\to\infty}\frac{1}{x^4}}{\lim_{x\to\infty}3-\lim_{x\to\infty}\frac{2}{x^4}}$$

$$= \frac{0+0}{3-0}$$

$$= \frac{0}{3} = 0$$

Interesting, hmm?

The following information is based on what we learned about rational functions back in algebra. Recall, the horizontal or oblique asymptotes of these functions depended on the degrees of the top and bottom.

THEOREM 2.7 End Behavior and Asymptotes of Rational Functions

Suppose $f(x) = \frac{p(x)}{q(x)}$ is a rational function, where

m = degree of top
n = degree of bottom

$$p(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_2x^2 + a_1x + a_0 \quad \text{and}$$
$$q(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_2x^2 + b_1x + b_0,$$

with $a_m \neq 0$ and $b_n \neq 0$.

a. **Degree of numerator less than degree of denominator** If $m < n$, then $\lim_{x\to\pm\infty} f(x) = 0$, and $y = 0$ is a horizontal asymptote of f.

b. **Degree of numerator equals degree of denominator** If $m = n$, then $\lim_{x\to\pm\infty} f(x) = a_m/b_n$, and $y = a_m/b_n$ is a horizontal asymptote of f.

c. **Degree of numerator greater than degree of denominator** If $m > n$, then $\lim_{x\to\pm\infty} f(x) = \infty$ or $-\infty$, and f has no horizontal asymptote.

d. **Slant asymptote** If $m = n + 1$, then $\lim_{x\to\pm\infty} f(x) = \infty$ or $-\infty$, and f has no horizontal asymptote, but f has a slant asymptote.

e. **Vertical asymptotes** Assuming f is in reduced form (p and q share no common factors), vertical asymptotes occur at the zeros of q.

The book uses "slant" instead of oblique. They also add a bonus about vertical asymptotes but it has nothing to do with these limits.

expl 5b: Redo the limit here by using this theorem.

$$\lim_{x\to\infty}\frac{2x+1}{3x^4-2}$$

degree on top = 1

degree on bottom = 4

Case a

$$\lim_{x\to\infty}\frac{2x+1}{3x^4-2} = 0 \text{ by Thm 2.7.}$$

The technique we developed in example 5*a* was *not* a complete waste of time. Consider these examples.

expl 6: Find the following limit.

$$\lim_{x\to-\infty}\frac{40x^4+x^2+5x}{\sqrt{64x^8+x^6}}$$

This is *not* a rational function. Do you know why? Divide *all* terms by $\sqrt{x^8}=x^4$ to get started here.

$$=\lim_{x\to-\infty}\frac{40x^4/x^4+x^2/x^4+5x/x^4}{\left(1/\sqrt{x^8}\right)\sqrt{64x^8+x^6}}$$

x^4 exponent was even

$$=\lim_{x\to-\infty}\frac{40+1/x^2+5/x^3}{\sqrt{64x^8/x^8+x^6/x^8}}$$

$$=\lim_{x\to-\infty}\frac{40+\underbrace{1/x^2}_{\to 0}+\underbrace{5/x^3}_{\to 0}}{\sqrt{64+\underbrace{1/x^2}_{\to 0}}}=\frac{40}{\sqrt{64}}=\frac{40}{8}=5$$

expl 7: Find the following limit.

$$\lim_{x\to-\infty}\frac{40x^3+x^2+5x}{\sqrt{64x^6+x}}$$

Here, we *should not* divide *all* by $\sqrt{x^6}=x^3$. If $x\to-\infty$ (which we assume), x^3 is negative while $\sqrt{x^6}$ is positive. To fix this, divide top by $-x^3$.

$$=\lim_{x\to-\infty}\frac{\frac{40x^3}{-x^3}+\frac{x^2}{-x^3}+\frac{5x}{-x^3}}{\sqrt{\frac{64x^6}{x^6}+\frac{x}{x^6}}}$$

x^3 exponent was odd

$$=\lim_{x\to-\infty}\frac{-40-\underbrace{\tfrac{1}{x}}_{\to 0}-\underbrace{5/x^2}_{\to 0}}{\sqrt{64+\underbrace{1/x^5}_{\to 0}}}=\frac{-40}{\sqrt{64}}=\frac{-40}{8}=-5$$

$$\sqrt{64x^6+x}\ \cancel{=8x^3}$$

A Word About Example 7:
This issue only arose because the limit assumed x was negative. We would *not* do anything special if the limit was "as $x \longrightarrow \infty$". Dividing by $\sqrt{x^6} = x^3$ on top and bottom would be fine as they are indeed equal when x is positive.

I will leave it to you to find the limit as $x \longrightarrow \infty$. You will see that you get a different answer.

$$\lim_{x\to\infty}\frac{40x^3+x^2+5x}{\sqrt{64x^6+x}} = \lim_{x\to\infty}\frac{\frac{40x^3}{x^3}+\frac{x^2}{x^3}+\frac{5x}{x^3}}{\sqrt{\frac{64x^6}{x^6}+\frac{x}{x^6}}} \dots$$

$$= \lim_{x\to\infty}\frac{40+\underbrace{1/x}_{\to 0}+\underbrace{5/x^2}_{\to 0}}{\sqrt{64+\underbrace{1/x^5}_{\to 0}}} = \frac{40}{\sqrt{64}} = \frac{40}{8} = 5$$

Interesting, this function has a different horizontal asymptote on the left as it has on the right. Again, this is *not* a rational function as the bottom is *not* a polynomial.

Graph the function, grasshopper.

Vertical Asymptotes:
You might be asked to find the vertical asymptotes of a rational function. Let's do a quick example to jog your memory.

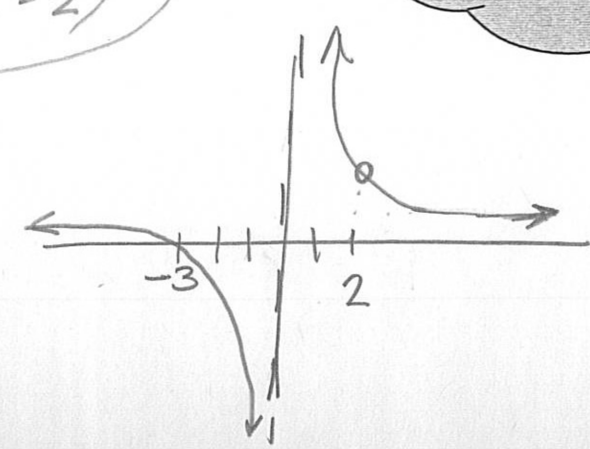
expl 8: Find the vertical asymptotes (VA) and holes if they exist for this rational function.

$f(x) = \frac{(x+3)\cancel{(x-2)}}{x\cancel{(x-2)}}$

hole occurs where
$x - 2 = 0$
$x = 2$

Cancel the common factor. Then, the VA occurs where the bottom is 0. The zeros of any cancelled factors are holes in the graph. Draw it now.

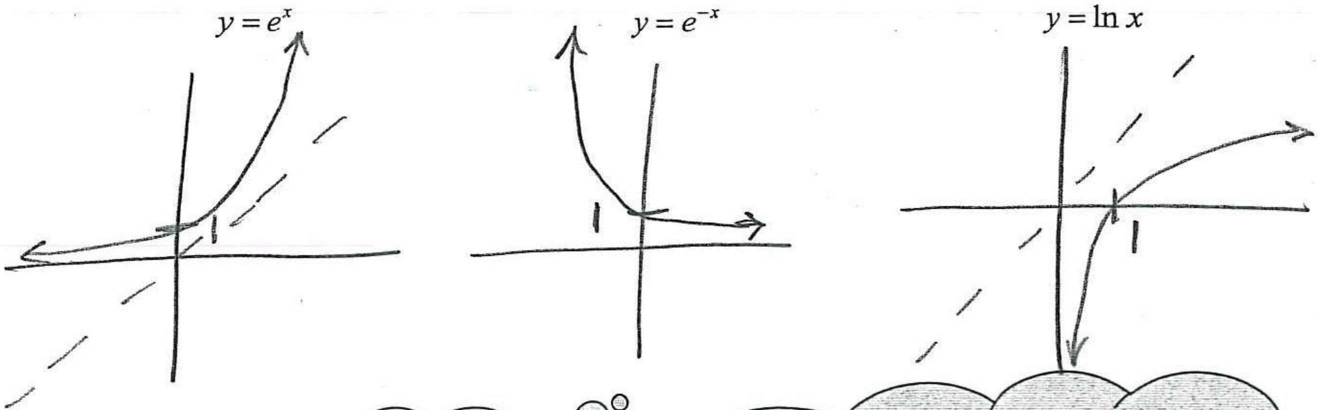
VA: $x = 0$
hole: $x = 2$



$f(x) = \frac{(x+3)(x-2)}{x(x-2)}$

Transcendental Functions:

The exponential functions and their inverses, the logarithmic functions, are transcendental. The trig functions we have been playing with are also. Can you draw these from memory?



Recall that e is an irrational number approximately equal to 2.72.

The graph of $y = e^{-x}$ can be thought of as a reflection of $y = e^{x}$ over the y-axis. The natural log function ($y = \ln x$) is the inverse of the exponential function $y = e^{x}$.

Fill in these limits as you see them from your fine graphs.

$\lim_{x \to -\infty} e^{x} = 0$ $\qquad \lim_{x \to -\infty} e^{-x} = \infty$ $\qquad \lim_{x \to 0^{+}} \ln x = -\infty$

$\lim_{x \to \infty} e^{x} = \infty$ $\qquad \lim_{x \to \infty} e^{-x} = 0$ $\qquad \lim_{x \to \infty} \ln x = \infty$

expl 9: Find the following limits.

a.) $\lim_{x \to \infty} (1 - \ln x) = \lim_{x \to \infty} 1 - \lim_{x \to \infty} (\ln x) = 1 - \infty = -\infty$

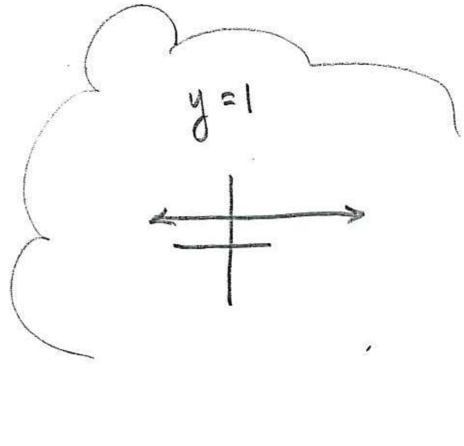
2.3 rule

2.3 rule

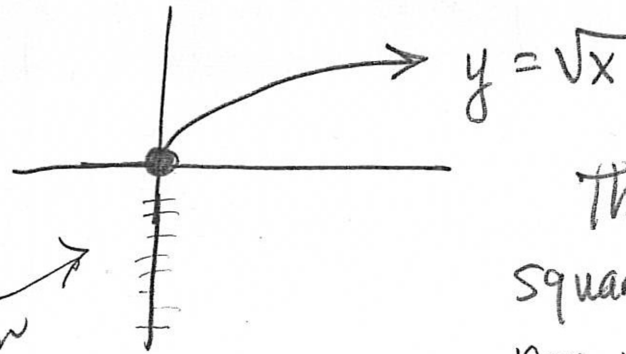
b.) $\lim_{x \to 0^{+}} (1 - \ln x) = \lim_{x \to 0^{+}} 1 - \lim_{x \to 0^{+}} (\ln x)$

$= 1 - -\infty$

$= \infty$



Addendum What is $y=\sqrt{x}$?



There are no points on this graph where $y<0$.

The output of the square root fnc. is non-negative (positive or zero).

Principal Square Root: $\sqrt{x}$ = the non-negative number I square to get x.

So, $\sqrt{9}=3$ because $3^2=9$.

It is a misconception that $\sqrt{9}=\pm 3$.

It comes from solving eqns like

$$x^2=9$$
$$\sqrt{x^2}=\sqrt{9}$$
$$|x|=3$$
$$x=\pm 3$$

The $\sqrt{x^2}$ is not just x. But rather $\sqrt{x^2}=|x|$.

To convince you that $\sqrt{x^2}\neq x$ But $\sqrt{x^2}=|x|$:

Let $x=-3$

then $(x^2)=(-3)^2=9$

and $\sqrt{9}$ is not -3 you started with. It's $|-3|$ which is $|x|$.