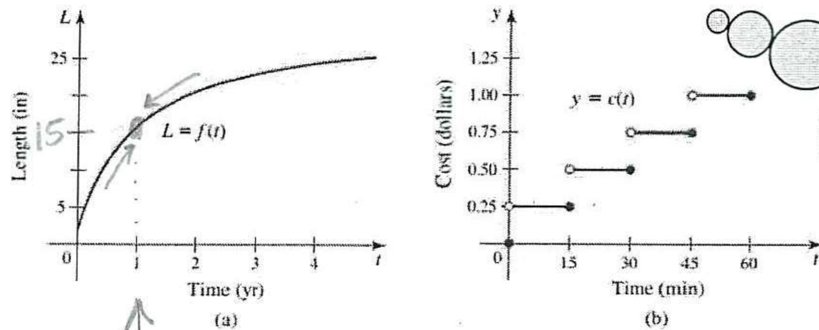
2:00

Can you trace a function's graph from the far left to the far right without lifting the pencil?

Calculus I
Class notes
Continuity (section 2.6)

The informal definition of a continuous function appears above. You might have seen that in algebra class. Consider these pictures here to better understand it. We will then move on to a more rigorous definition that uses limits.



The left graph is continuous. Trace it from left to right completely. The right graph is *not* continuous.

Figure 2.42

$t=1$

Definition: Continuity at a Point: A function is said to be continuous at $x=a$ if and only if $\lim_{x\to a} f(x) = f(a)$.

More specifically, this means three things.

1.) $f(a)$ is defined. This means that a is in the domain of f.

2.) $\lim_{x\to a} f(x)$ exists

3.) $\lim_{x\to a} f(x) = f(a)$

Do you recall how we use right and left-sided limits to find a limit?

Hence, if f is continuous at a, then we can use direct substitution to find this limit.

expl 1a: Approximate $\lim_{t\to 1} f(t)$ for the continuous graph of $L = f(t)$ shown on the left above.

$\lim_{t\to 1} f(t) = 15$ because $\lim_{t\to 1^-} f(t) = \lim_{t\to 1^+} f(t) = 15$

expl 1b: Try to find $\lim_{t\to 15} c(t)$ for the *non*-continuous graph of $y = c(t)$ shown on the right above.

Just try, I dare ya!

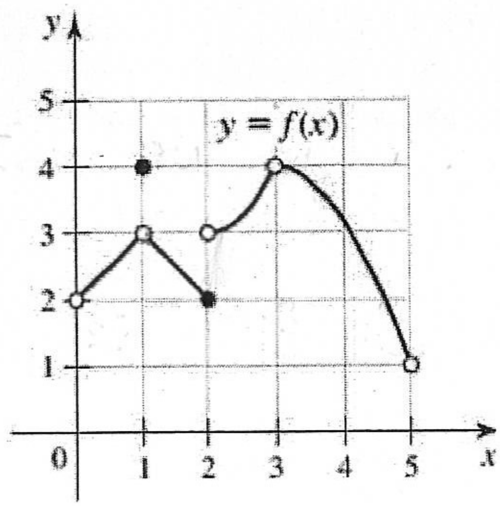
$\lim_{t\to 15^-} c(t) = 0.25$ But $\lim_{t\to 15^+} c(t) = 0.50$

So, $\lim_{t\to 15} c(t)$ dne.

Definition: Point of discontinuity: An x-value where the graph of the function is *not* continuous. You could *not* trace the function without lifting your pencil at this x-value.

expl 2: Find the points of discontinuity in the interval (0, 5). For each point of discontinuity, state which of the three criteria from page 1 is *not* met.

Notice how the interval (0, 5) does *not* include the endpoints.



Point of Discontinuity (x-value)	Which criteria is *not* met?
$x = 1$	3.) $\lim_{x\to 1} f(x) \neq f(1)$.
$x = 2$	2.) $\lim_{x\to 2} f(x)$ dne $\left(\lim_{x\to 2^-} f \neq \lim_{x\to 2^+} f\right)$
$x = 3$	1.) $f(3)$ dne

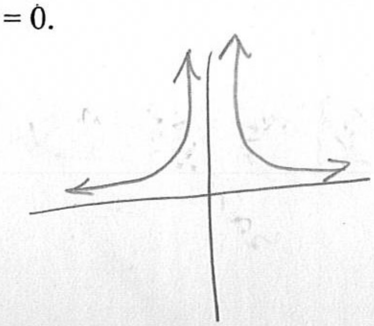
Classification of Points of Discontinuity:
We further break down points of discontinuity into three types.

Definition: Removable discontinuity: This is a point of discontinuity where you could remove the discontinuity by redefining a function value. This would be true for $x = 1$ and $x = 3$ above.

Definition: Jump discontinuity: This is a point of discontinuity where the graph "jumps" to a new location. It is *not* removable like above. Where does this happen in the above graph?

$X = 2$

Definition: Infinite discontinuity: This is a point of discontinuity $x = a$ because $\lim_{x\to a} f(x)$ is equal to infinity or its negative (and is said to *not* exist). Picture the graph of $y = 1/x^2$ and its point of discontinuity $x = 0$.



$y = \frac{1}{x^2}$

Algebraic Analysis of Continuity:

We will check the three criteria given on page 1 one at a time.

expl 3: Determine if the function is continuous at a.

a.) $f(x)=\dfrac{2x^2+3x+1}{x^2+5x}$, $a=-5$

Criteria #1: $f(-5)=\dfrac{2(-5)^2+3(-5)+1}{(-5)^2+5(-5)}=\dfrac{36}{0}$ $f(-5)$ dne

So, $f(x)$ is not continuous at $x=-5$.

b.) $f(x)=\begin{cases}\dfrac{x^2-4x+3}{x-3}, & x\neq 3\\ 2, & x=3\end{cases}$ for $a=3$

Use the first formula to find $\lim\limits_{x\to 3} f(x)$. Is it equal to $f(3)$?

Criteria #1: $f(3)=2$ ✓

Criteria #2: $\lim\limits_{x\to 3}\dfrac{x^2-4x+3}{x-3}$

$=\lim\limits_{x\to 3}\dfrac{\cancel{(x-3)}(x-1)}{\cancel{(x-3)}}=\lim\limits_{x\to 3}(x-1)=3-1=2$ ✓

Criteria #3: Is $\lim\limits_{x\to 3} f(x)=f(a)$?

yes $\rightarrow$ So, the fnc is continuous at $x=3$.

c.) $f(x)=\dfrac{5x-2}{x^2-9x+20}$, $a=4$

Be ready to factor to explore functions.

Criteria #1: $f(4)=\dfrac{5\cdot 4-2}{4^2-9\cdot 4+20}=\dfrac{18}{0}$

$f(4)$ dne

Criteria #1 fails

$\rightarrow$ So, $f(x)$ is not continuous at $x=4$.

Continuity Rules:

These can be handy when exploring functions related to ones you already know.

> **THEOREM 2.8 Continuity Rules** $c \in \mathbb{R}$ $\quad n \in \mathbb{Z}^+$
>
> If f and g are continuous at a, then the following functions are also continuous at a. Assume c is a constant and $n > 0$ is an integer.
>
> **a.** $f + g$ **b.** $f - g$
>
> **c.** cf **d.** fg
>
> **e.** f/g, provided $g(a) \neq 0$ **f.** $(f(x))^n$

These can be proven using limit laws.

The above rules can be used to show that the following are true.

> **THEOREM 2.9 Polynomial and Rational Functions**
>
> **a.** A polynomial function is continuous for all x.
>
> **b.** A rational function (a function of the form $\frac{p}{q}$, where p and q are polynomials) is continuous for all x for which $q(x) \neq 0$.

Find the points of discontinuity of a rational function by solving "bottom = 0".

Do you recall the definition of a composite function? We have some important rules here.

> **THEOREM 2.10 Continuity of Composite Functions at a Point**
>
> If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ is continuous at a.

Pronounced "f of g of x".

> **THEOREM 2.11 Limits of Composite Functions**
>
> **1.** If g is continuous at a and f is continuous at $g(a)$, then
>
> $$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right). = f(g(a))$$
>
> **2.** If $\lim_{x \to a} g(x) = L$ and f is continuous at L, then
>
> $$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right). = f(L)$$

I will add to what the book has here by noting that statement 1 implies $\lim_{x \to a} f(g(x)) = f(g(a))$.

Likewise, statement 2 implies $\lim_{x \to a} f(g(x)) = f(L)$ given its definition of L.

These theorems can be used to find limits such as $\lim_{x \to 2} \left(\frac{x^2+3}{x+4}\right)^3$. Do you see the composite function? Find f and g such that $(f \circ g)(x) = \left(\frac{x^2+3}{x+4}\right)^3$.

$\longrightarrow f(x) = x^3$ and $g(x) = \left(\frac{x^2+3}{x+4}\right)$

12:00

$f(x) = \sqrt{x}$ $\qquad g(x) = \dfrac{x^3-2x^2-8x}{x-4}$

expl 4: Find the following limit. Leave your answer in exact (radical) form.

$$\lim_{x\to 4}\sqrt{\frac{x^3-2x^2-8x}{x-4}}$$

Define f and g such that $f(g(x)) = \sqrt{\dfrac{x^3-2x^2-8x}{x-4}}$.

Use statement 2 of Theorem 2.11 to think your way through this limit.

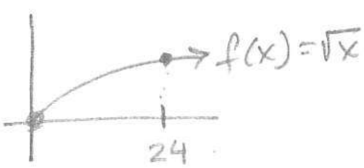
First, find $\lim\limits_{x\to 4} g(x) = \lim\limits_{x\to 4} \dfrac{x^3-2x^2-8x}{x-4}$

$= \lim\limits_{x\to 4} \dfrac{x(x^2-2x-8)}{(x-4)}$

$= \lim\limits_{x\to 4} \dfrac{x\cancel{(x-4)}(x+2)}{\cancel{(x-4)}}$

$= \lim\limits_{x\to 4} x(x+2) = 4\cdot 6 = 24 = L$

Is f continuous at L = 24? — yes - it fits the defn. on page 1.



So, by Theorem 2.11, Statement # 2,

$\lim\limits_{x\to 4}\sqrt{\dfrac{x^3-2x^2-8x}{x-4}} = f(L) = f(24) = \sqrt{24}$

$= \sqrt{4\cdot 6}$

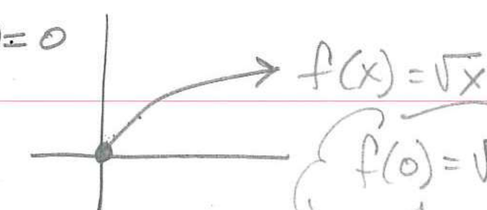
$= 2\sqrt{6}$

Definition: Continuity on an Interval's Endpoint (Right-continuous): A function f is continuous from the right (right-continuous) at a if $\lim\limits_{x\to a^+} f(x) = f(a)$.

Draw the graph of $f(x) = \sqrt{x}$ and use it to find $\lim\limits_{x\to 0^+} f(x) = 0$

Is $f(x) = \sqrt{x}$ right-continuous at 0?

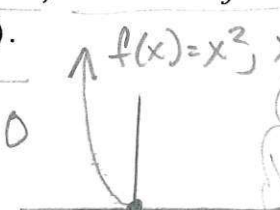
yes because $\lim\limits_{x\to 0^+} f(x) = f(0)$



$f(0) = \sqrt{0} = 0$

Definition: Continuity on an Interval's Endpoint (Left-continuous): A function f is continuous from the left (left-continuous) at a if $\lim\limits_{x\to a^-} f(x) = f(a)$.

Draw the graph of $f(x) = x^2,\ x \le 0$ and use it to find $\lim\limits_{x\to 0^-} f(x) = 0$



$f(0) = 0^2 = 0$

This means f is the familiar parabola but with a **restricted domain**; only graph the left half.

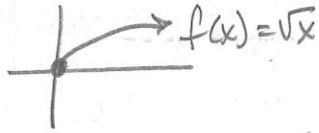
Is $f(x) = x^2,\ x \le 0$ left-cont at $x = 0$?

— yes because $\lim\limits_{x\to 0^-} f(x) = f(0)$.

Definition: Continuity on an Interval: A function f is continuous on an interval I if it is continuous on all points in the interval I.

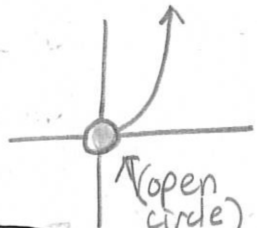
If I contains endpoints (meaning its notation is written with square brackets and *not* just parentheses), continuity on I means continuity from the left *or* continuity from the right.

Once again draw $f(x)=\sqrt{x}$. Determine if f is continuous on the interval $[0, \infty)$.



Since $f(x)=\sqrt{x}$ is right-cont at 0, we will say f is continuous on $[0,\infty)$.

Draw $f(x)=x^2$, ~~$x<0$~~ $x>0$. Determine if f is continuous on the interval $[0, \infty)$. Is it continuous on $(0, \infty)$?



Is $f(x)=x^2$, $x>0$ continuous on the interval $[0,\infty)$?
– No because 0 is not in the domain of the fnc. and so $\lim_{x\to 0^+} f(x) \neq f(0)$.

Is f cont on $(0,\infty)$? – yes

The **restricted domain** changed from the last page. How does that affect the graph?

Our knowledge about composite functions (Theorem 2.10) leads us to this rule.

> **THEOREM 2.12 Continuity of Functions with Roots**
> Assume n is a positive integer. If n is an odd integer, then $(f(x))^{1/n}$ is continuous at all points at which f is continuous.
> If n is even, then $(f(x))^{1/n}$ is continuous at all points a at which f is continuous and $f(a) > 0$.

Again, this restriction about roots hinges on if the root's index (n) is even. If we are dealing with a square root, fourth root, etc. we must ensure $f(a) > 0$. Otherwise, $(f(a))^{1/n}$ would *not* exist (in the real numbers).

Picture any function f, for instance $f(x)=x+4$. In the language of the theorem, we might be investigating $(f(x))^{1/5}=(x+4)^{1/5}=\sqrt[5]{x+4}$.

$n=5$

$\boxed{2{:}00}$

$f(x) = x^2 - 3x + 2$

$n = 2$

expl 5: Determine the intervals where g is continuous. At which finite endpoints is g continuous from left or right?

a.) $g(x) = \sqrt{x^2 - 3x + 2}$ $(f(x))^{1/n}$

In the theorem's language, determine the function f and the value of n.

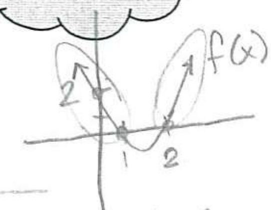
From Thm 2.12, we need to find where $f(x) = x^2 - 3x + 2$ is continuous and $f(a) > 0$. We know f is cont on $(-\infty, \infty)$ because it's a polynomial. Find $f(a) > 0$:

$f(x) = x^2 - 3x + 2 = 0$

$(x-2)(x-1) = 0$

$x = 2,\ x = 1$

Where is $f > 0$?

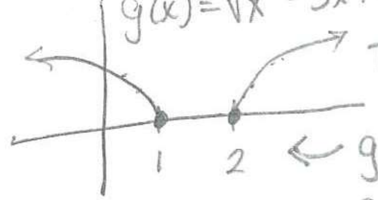


So, from our graph, we see $f(x) > 0$ on $(-\infty, 1)$ and $(2, \infty)$.

So, by thm 2.12, $g(x) = (x^2 - 3x + 2)^{1/2}$ is continuous on $(-\infty, 1)$ and $(2, \infty)$.

The finite endpoints $x = 1$ and $x = 2$ must be looked at.

$g(x) = \sqrt{x^2 - 3x + 2}$



To verify $g(x)$ hits x-axis:

$g(1) = \sqrt{1 - 3 + 2} = \sqrt{0} = 0$

$g(2) = \sqrt{4 - 6 + 2} = \sqrt{0} = 0$

From graph and accompanying algebra, we see $\lim_{x \to 1^-} g = g(1)$ and $\lim_{x \to 2^+} g = g(2)$, we see g is left-cont at $x = 1$ and right-cont at $x = 2$. Hence, g is continuous at $(-\infty, 1]$ and $[2, \infty)$.

b.) $g(x) = \sqrt[3]{x^2 - 3x + 2}$ $= (f(x))^{1/n}$

$f(x) = x^2 - 3x + 2$

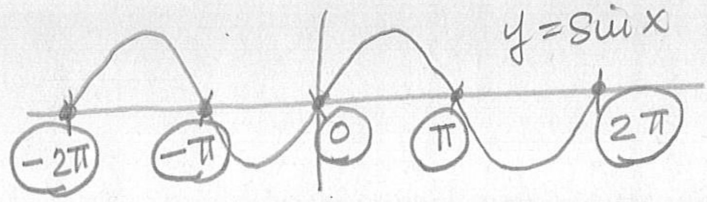
$n = 3$ (odd)

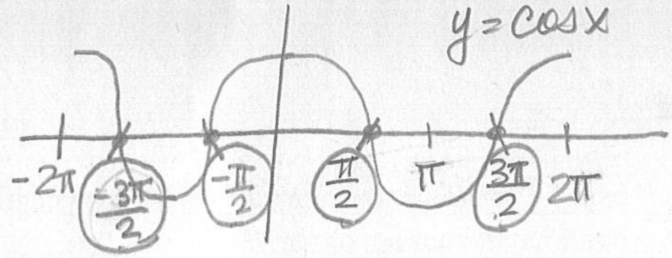
Thm 2.12 says g is continuous where f is continuous.

And f is continuous on $(-\infty, \infty)$.

Hence, $g = \sqrt[3]{x^2 - 3x + 2}$ is continuous on $(-\infty, \infty)$.

You could graph to check.





Continuity of Transcendentals: Trigonometry:

Picture the graphs of $y = \sin(x)$ and $y = \cos(x)$. Would you agree that they are both continuous over the interval $(-\infty, \infty)$? Sketch them a bit here.

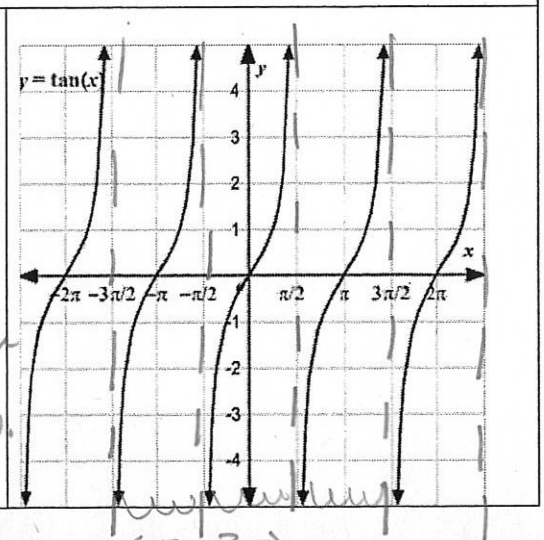
Theorem 2.8, part *e*, is used to show that the other main trig functions are continuous at a, as long as they are defined at a. Consider the following definitions of these trig functions and think on what values make the bottoms 0 (which would be those values for which the function is *not* defined).

$y = \tan(x) = \frac{\sin(x)}{\cos(x)}$ This would be undefined (so not continuous) at ... $-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$ or $x = \frac{\pi}{2} + n\pi$ for $n \in \mathbb{Z}$.	cosecant $y = \csc(x) = \frac{1}{\sin(x)}$ This would be not continuous at ... $-2\pi, -\pi, 0, \pi, 2\pi, \ldots$ or $x = n\pi$ for $n \in \mathbb{Z}$.
secant $y = \sec(x) = \frac{1}{\cos(x)}$ This would be not continuous at $x = \frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$.	cotangent $y = \cot(x) = \frac{\cos(x)}{\sin(x)}$ This would be not continuous at $x = n\pi$ for $n \in \mathbb{Z}$.

Here is a graph of the tangent function I got from https://www.varsitytutors.com.

You can see where the function is undefined (vertical asymptotes) are exactly the points of discontinuity we predicted above.

$y = \tan x$ is only continuous between these vertical asymptotes.



or

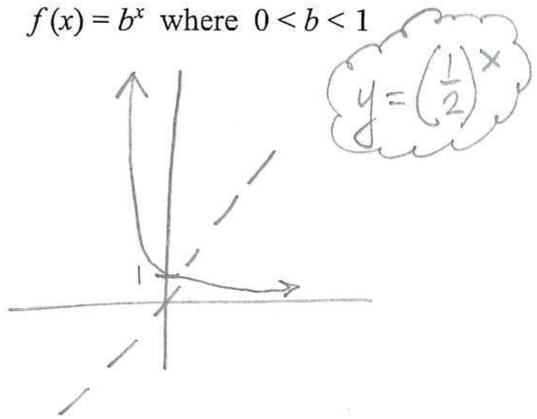
$\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right)$ and $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

Continuity of Transcendentals: Exponentials and Logarithms:

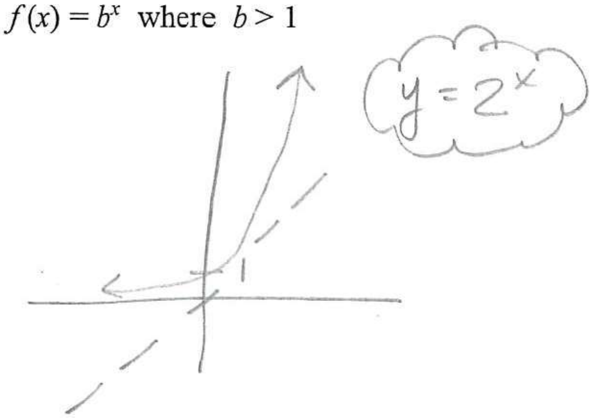
We next consider the exponential function $f(x) = b^x$ where $b > 0$ and $b \neq 1$. As you will recall, its domain is $(-\infty, \infty)$ and it is continuous along its entire domain.

Draw an example of this function where $b < 1$ and one where $b > 1$ for your reference.

$f(x) = b^x$ where $0 < b < 1$



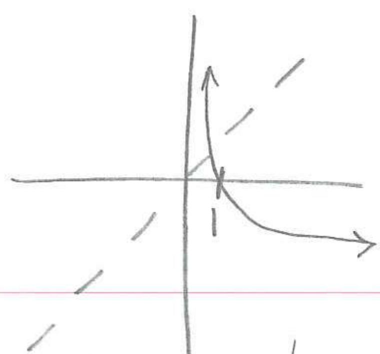
$f(x) = b^x$ where $b > 1$



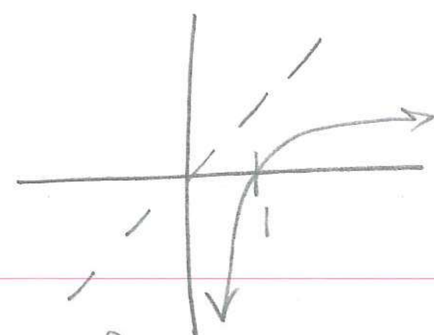
Recall, the inverse of the exponential function is the logarithmic function. We also see that these functions are continuous over their entire domains.

Draw an example of this function where $b < 1$ and one where $b > 1$ for your reference.

$f^{-1}(x) = \log_b x$ where $0 < b < 1$



$f^{-1}(x) = \log_b x$ where $b > 1$



domain: $(0, \infty)$

What is the domain of these log functions? It will be the same as the range of the exponential functions from whence they came.

It happens to be true that a continuous function's inverse will also be continuous.

These log. fncs are continuous on the interval $(0, \infty)$.

Intermediate Value Theorem (IVT):

This is a surprisingly handy fact. It says that if a function is continuous in some interval, then the graph must contain at least one point for each y-value inside the range of that interval. Here is a picture from the book.

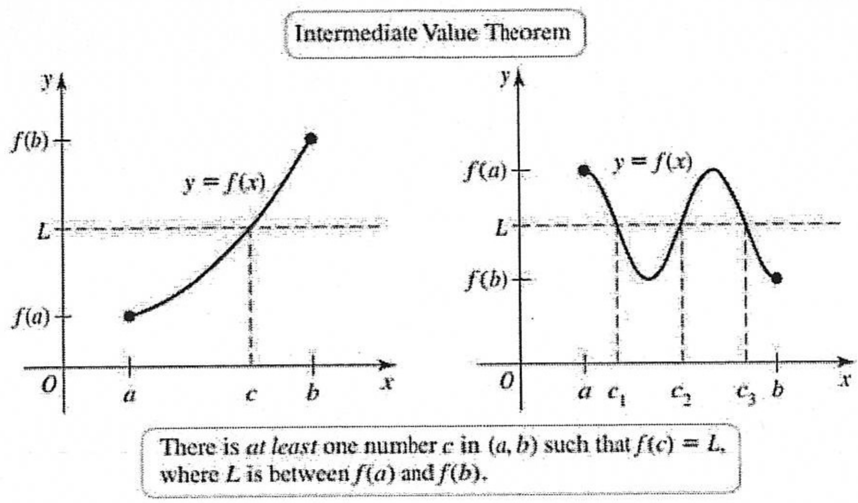


Figure 2.51

Here is the statement in its full glory.

y value

> **THEOREM 2.14 Intermediate Value Theorem**
> Suppose f is continuous on the interval $[a, b]$ and L is a number strictly between $f(a)$ and $f(b)$. Then there exists at least one number c in (a, b) satisfying $f(c) = L$.

x value

The book shows us why the function must be continuous for this statement. See the example here that is *not* continuous and so does *not* necessarily have such a point.

We will use the theorem on the next page.

This was more useful before the advent of graphing calculators. However, you will still see this theorem used in higher math.

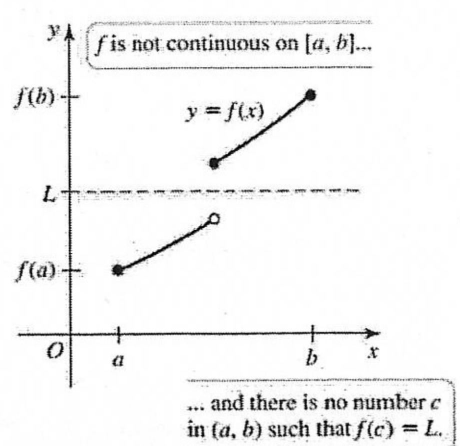


Figure 2.52

L

expl 6: Use the Intermediate Value Theorem to show the following equation has a solution in the given interval. Use a grapher to find the solutions, rounding to three decimal places. Lastly, graph the function on paper.

$x^3 - 5x^2 + 2x = -1, \quad (-1, 5)$

From the theorem, determine a and b. Find $f(a)$ and $f(b)$. Is the premise of the theorem satisfied? Can you use it?

$f(x) = x^3 - 5x^2 + 2x \qquad a = -1, \quad b = 5$

We know $f(x)$ is continuous on $[-1, 5]$.

And $f(a) = f(-1) = (-1)^3 - 5(-1)^2 + 2(-1)$

$= -1 - 5 - 2$

$= -8$

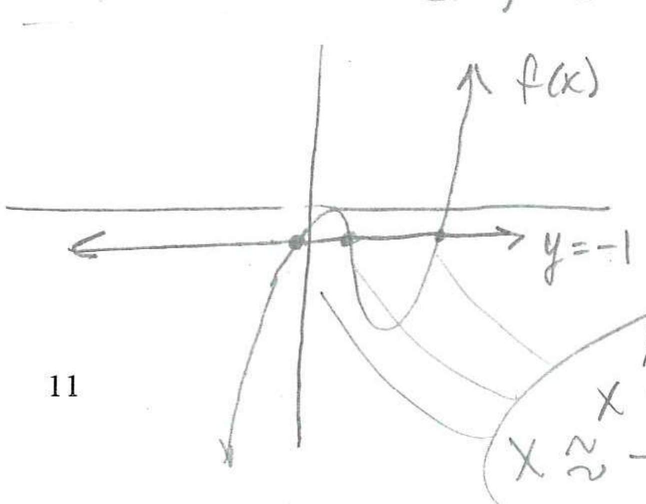
and $f(b) = f(5) = 5^3 - 5 \cdot 5^2 + 2 \cdot 5$

$= 125 - 125 + 10$

$= 10$

Let $L = -1$ (from eqn) and we see how L is strictly between $f(a) = -8$ and $f(b) = 10$.

So, theorem 2.14 says there exists at least one number c in $(-1, 5)$ satisfying $f(c) = L = -1$



$x \approx 4.507$

$x \approx 0.778$

$x \approx -0.285$