2:00

We could always use limits to find derivatives. In practice, we will use shortcut rules instead.

Calculus I
Class notes
Rules of Differentiation (section 3.3)

It is possible to use the limit definition for derivatives and it is good to practice that. Limits are an important topic and this gives us ample opportunity to work with them. However, in practice, we will *not* use it to find the derivative of a function. We have many rules, some specific and some general, that we will see through the next sections to shortcut our work.

The book does provide proofs for many of these rules but we will skip those for now. Here are the rules we will work with in this section.

THEOREM 3.2 Constant Rule
If c is a real number, then $\frac{d}{dx}(c) = 0$.

$y=4$ $\qquad \frac{d}{dx}(4) = 0$

THEOREM 3.3 Power Rule
If n is a nonnegative integer, then $\frac{d}{dx}(x^n) = nx^{n-1}$.

We will see later that this rule is true for $n \in \mathbb{R}$.

$n \in \{0, 1, 2, 3, \ldots\}$

$\frac{d}{dx}(x^3) = 3x^{3-1} = 3x^2$

THEOREM 3.4 Constant Multiple Rule
If f is differentiable at x and c is a constant, then

$f'(x)$ exists

$$\frac{d}{dx}(cf(x)) = cf'(x).$$

$\frac{d}{dx}(2x^3) = 2 \cdot \frac{d}{dx}(x^3) = 2 \cdot 3x^2 = 6x^2$

THEOREM 3.5 Sum Rule
If f and g are differentiable at x, then

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x).$$

This, as you might imagine, works for subtraction too.

Handout: Common Derivatives and Integrals:
Paul Dawkins, my internet hero, provides this which is another good one to tuck in your folder. You can find many such references online.

$a^0 = 1$, if $a \neq 0$
Zero Exponent Rule.

Exponential Functions:

Recall that e is the irrational number that is roughly equal to 2.72. The function $f(x) = e^x$ is a common construct that pops up in many problems. A fun little tidbit about this **natural exponential function** is that it is its own derivative. Rather,

$$\frac{d}{dx}\left(e^x\right) = e^x.$$

Easiest rule you'll ever get.

Let's do some examples.

expl 1: Use the rules to find the following derivatives. You can label your answers using prime notation (like $f'(x)$). For reasons left 'til later, you are told that $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$.

a.) $f(x) = 5$

Constant rule

$f'(x) = 0$ OR $\frac{d}{dx}(5) = 0$

b.) $f(x) = 5x^3 \rightarrow \frac{d}{dx}(5x^3)$

Power rule

Constant Multiple rule

$= 5 \cdot \frac{d}{dx}(x^3)$

$= 5 \cdot 3x^2 = 15x^2$

$f'(x) = 15x^2$

c.) $f(t) = 5\sqrt{t} \rightarrow \frac{d}{dt}(5\sqrt{t})$

$\frac{d}{dt}(\sqrt{t}) = \frac{1}{2\sqrt{t}}$

Constant Multiple Rule

$= 5\frac{d}{dt}(\sqrt{t})$

(Given)

$= 5 \cdot \frac{1}{2\sqrt{t}} = \boxed{\frac{5}{2\sqrt{t}} = f'(t)}$

Just go term by term.

d.) $f(x) = \frac{5}{3}x^3 + 6x^2 - \frac{2}{3}x^1 + 7$

Sum Rule
Power Rule
Constant Multiple Rule
Constant Rule

$f'(x) = \frac{5}{3}\cdot 3x^{3-1} + 6 \cdot 2x^{2-1} - \frac{2}{3} \cdot 1x^{1-1} + 0$

$= 5x^2 + 12x - \frac{2}{3}x^0$ $x^0 \rightarrow 1$

$f'(x) = 5x^2 + 12x - \frac{2}{3}$

e.) $g(x) = -3x^3 - 4x$

$g'(x) = -3 \cdot 3x^2 - 4$

$= -9x^2 - 4$

$\frac{d}{dx}(5x^3) = \underline{15x^2}$ (part b)

expl 2: Use the result from example 1 to evaluate the following.

$\frac{d}{dx}(5x^3)\Big|_{x=2} = 15 \cdot 2^2 = 15 \cdot 4 = 60$

The slope of the tangent line to $y = 5x^3$ at $x = 2$ is 60.

expl 3: The position of a small rocket launched upward is given by $s(t) = -5t^2 + 40t + 100$ where t is the time in seconds after launch $(0 \le t \le 10)$ and $s(t)$ is in meters above the ground.

a.) Find the instantaneous velocity (rate of change in position) for $0 \le t \le 10$.

$v(t) = s'(t) = -10t + 40$

b.) At what time is the instantaneous velocity zero?

$t = ?$ $\quad v(t) = 0$

$v(t) = -10t + 40$

$0 = -10t + 40$

$-40 = -10t$

$t = 4$ seconds

For the most part, you are expected to answer questions algebraically.

c.) How fast is the rocket going when it first reaches a height of 175 meters?

$v(t) = ?$ $\quad s(t) = 175$

$s(t) = -5t^2 + 40t + 100$

$175 = -5t^2 + 40t + 100$

$0 = -5t^2 + 40t - 75$

$0 = -5(t^2 - 8t + 15)$

$0 = -5(t-3)(t-5)$

$t - 3 = 0$ or $t - 5 = 0$

$t = 3$ $\quad \cancel{t = 5}$

$v(3) = -10(3) + 40$

$= 10$ m/s

Can you graph without your calculator?

d.) Graph $s(t) = -5t^2 + 40t + 100$.
Find the points of interest for parts *b* and *c*.

y-int: $s(0) = 100$

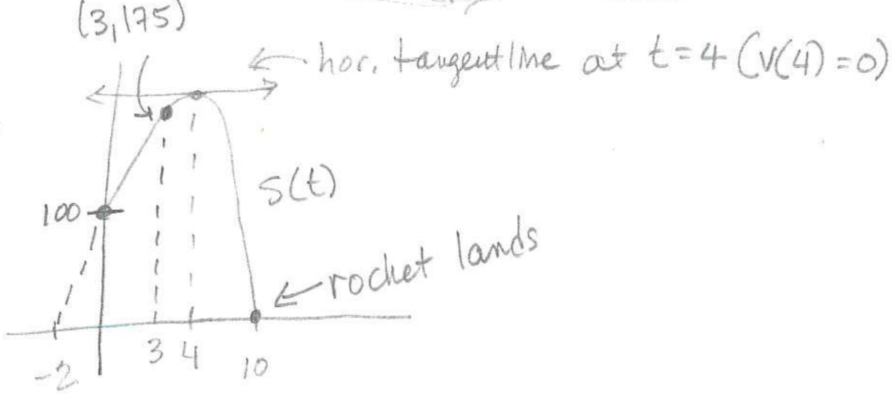
t-int: $0 = -5t^2 + 40t + 100$

$0 = -5(t^2 - 8t - 20)$

$0 = -5(t+2)(t-10)$

$t + 2 = 0$ or $t - 10 = 0$

$t = -2$ $\quad t = 10$



expl 4: Find the derivative of the following by expanding or simplifying the expression.

$h(x) = \left(\sqrt{x}+1\right)\left(\sqrt{x}-1\right)$

FIOL

$h(x) = (\sqrt{x})^2 + \sqrt{x} - \sqrt{x} - 1$

$h(x) = x - 1$

$h'(x) = 1$

We will learn rules that can handle this directly. The lesson here is that they are *not* always needed.

Higher-Order Derivatives:

Since the derivative of a function is itself a function, we could find *its* derivative. In other words, we can find the derivative of the derivative, called the second derivative. And so on and so on. We can find the third derivative, fourth derivative, etc. for as long as they are defined.

As you see here, we will use prime notation to denote the first few derivatives. Take a look at the notation for higher derivatives.

> DEFINITION **Higher-Order Derivatives**
>
> Assuming $y = f(x)$ can be differentiated as often as necessary, the **second derivative** of f is
>
> $$f''(x) = \frac{d}{dx}(f'(x)).$$
>
> For integers $n \geq 1$, the **nth derivative** of f is
>
> $$f^{(n)}(x) = \frac{d}{dx}(f^{(n-1)}(x)).$$

$f''(x)$

We answer the question, "How fast is the derivative function changing?"

expl 5: Find the first, second, third, and fourth derivatives for this function.

From pg 2: $\frac{d}{dp}(e^p) = e^p$

$g(p) = \frac{5}{3}p^3 + 6e^p$

$g'(p) = \frac{5}{\cancel{3}} \cdot \frac{\cancel{3}}{1} p^2 + 6e^p = \boxed{5p^2 + 6e^p = g'(p)}$

$g''(p) = 10p^1 + 6e^p$

$g'''(p) = 10 \cdot 1 p^{1-1} + 6e^p = \boxed{10 + 6e^p = g'''(p)}$

$g^{(4)}(p) = 6e^p$