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We have a rule for the sum or difference of two functions. What about when we multiply or divide two functions?

Calculus I
Class notes
Product and Quotient Rules for Differentiation (section 3.4)

Keep in mind that the derivative of a function is its rate of change.

Consider a person running down the street. Their running speed would be equal to the product of their stride length and their stride rate. For instance, if your stride length is three feet per stride and your stride rate is two strides per second, what is your running speed? Include units.

running speed = stride length * stride rate = $\frac{3 \text{ ft}}{\cancel{\text{stride}}} * \frac{2 \cancel{\text{strides}}}{\text{second}} = 6$ ft/sec.

Now, suppose your stride length increases to 3.5 feet per stride while your stride rate remains at two strides per second, what is your running speed? Include units.

running speed = stride length * stride rate = $\frac{3.5 \text{ ft}}{\cancel{\text{stride}}} * \frac{2 \cancel{\text{strides}}}{\text{sec}} = 7$ ft/sec

We could calculate the **change in speed** by subtracting the two answers above. However, we could also multiply the *change* in stride length by the stride rate. Do this now, including units.

change in speed = change in stride length * stride rate = $\frac{0.5 \text{ ft}}{\cancel{\text{stride}}} * \frac{2 \cancel{\text{stride}}}{\text{sec}} = 1$ ft/sec

Alternatively, imagine your stride length is still three feet per stride but your stride rate increases to 2.25 strides per second. Find your change in speed by multiplying the stride length by the *change* in stride rate. Include units.

Change in speed = stride length * change in stride rate = $\frac{3 \text{ ft}}{\cancel{\text{stride}}} * \frac{0.25 \cancel{\text{stride}}}{\text{sec}} = 0.75$ ft/sec

We have seen the result of changing one of the factors. But what happens if we change *both* stride length *and* stride rate? You can expect a change in speed of $1 + 0.75 = 1.75$ feet per second.

This leads us to the Product Rule. As we look at it on the next page, imagine $f(x)$ to be stride length and $g(x)$ to be stride rate. To find the rate of change (in other words, the derivative) in running speed (which is the product of $f(x)$ and $g(x)$), we have to do some maneuvering.

$a^n \cdot a^m = a^{n+m}$

THEOREM 3.6 Product Rule
If f and g are differentiable at x, then

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

We will use it to find the derivatives of products of functions.

expl 1: Go nuts. Find these derivatives using the new rule.

a.) $h(x) = 3x^4(2x^2 - 1)$ $\quad f(x) = 3x^4,\ g(x) = 2x^2 - 1$

$h'(x) = 12x^3(2x^2-1) + 3x^4(4x)$

$= 24x^5 - 12x^3 + 12x^5$

$h'(x) = 36x^5 - 12x^3$

$f(x) = 3x^4$

$f'(x) = 3 \cdot 4x^{4-1}$

$= 12x^3$

$g(x) = 2x^2 - 1$

$g'(x) = 2 \cdot 2x^{2-1} - 0$

$= 4x$

b.) $f(t) = t^{5/3}e^t$ $\quad$ 1st func: $t^{5/3}$; 2nd func: e^t

$f'(t) = \frac{5}{3}t^{2/3} \cdot e^t + t^{5/3} \cdot e^t$

OR

$f'(t) = e^t t^{2/3}\left(\frac{5}{3} + t\right)$

$\frac{5}{3} - 1 = \frac{5}{3} - \frac{3}{3} = \frac{2}{3}$

$t^{5/3} = t^{2/3} \cdot t^{3/3} = t^{2/3} \cdot t$

The book now tells us that the Power Rule (from the last section) works for any $n \in \mathbb{R}$.

$\frac{d}{dx}(x^n) = nx^{n-1}$

Sort of. If n is irrational, then the Power Rule holds only when $x > 0$.

like π or e or $\sqrt{2}$

$\frac{d}{dx}(x^\pi) = ?$ can be done with Power Rule but restrict $x > 0$.

If n is rational or rather $n = p/q$ where $p, q \in \mathbb{Z}$ with *no* common factors and q is even, then the Power Rule holds only when $x > 0$.

$\frac{d}{dx}(x^{3/2}) = \frac{3}{2}x^{1/2}$ but $x > 0$.

Try to factor the top

$x^3 - 4x^2 + x$

$= x(x^2 - 4x + 1)$ never mind

(2:00

We see this expansion of the Power Rule because of the Quotient Rule. The book does not like to provide a rule *without* proof. With this Quotient Rule, they can now justify this Power Rule expansion. The book does an awesome job showing why the Quotient Rule works because of the Product Rule. I provide it here without proof.

THEOREM 3.7 Quotient Rule $f'(x)$ and $g'(x)$ exist

If f and g are differentiable at x and $g(x) \neq 0$, then the derivative of f/g at x exists and

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$

$$\frac{LoD(Hi) - HiD(Lo)}{(Lo)^2}$$

expl 2: Find the derivatives of these quotients. If it helps, rename the given function as $h(x)$.

a.) $f(x) = \dfrac{x^3 - 4x^2 + x}{x - 2}$

$$f'(x) = \frac{(x-2)(3x^2 - 8x + 1) - (x^3 - 4x^2 + x)(1)}{(x-2)^2}$$

$$= \frac{3x^3 - 8x^2 + x - 6x^2 + 16x - 2 - x^3 + 4x^2 - x}{(x-2)^2}$$

$$f'(x) = \frac{2x^3 - 10x^2 + 16x - 2}{(x-2)^2} = \cancel{\frac{2x^2(x-5) + 2(8x-1)}{(x-2)^2}}$$

$\frac{d}{dx}(x^n) = nx^{n-1}$

b.) $g(w) = \dfrac{\sqrt{w} + w}{\sqrt{w} - w}$

$a^n \cdot a^m = a^{n+m}$; $w^0 = 1$ $(w \neq 0)$

top fnc $= w^{1/2} + w$

deriv $= \frac{1}{2}w^{-1/2} + 1$

bottom fnc $= w^{1/2} - w$

deriv $= \frac{1}{2}w^{-1/2} - 1$

$$g'(w) = \frac{(w^{1/2} - w)(\frac{1}{2}w^{-1/2} + 1) - (w^{1/2} + w)(\frac{1}{2}w^{-1/2} - 1)}{(w^{1/2} - w)^2}$$

FOIL

$$= \frac{\cancel{\tfrac{1}{2}} - \frac{1}{2}w^{1/2} + w^{1/2} \cancel{- w} \cancel{- \tfrac{1}{2}} - \frac{1}{2}w^{1/2} + w^{1/2} \cancel{+ w}}{(w^{1/2} - w)^2}$$

$$= \frac{w^{1/2}}{(w^{1/2} - w)^2}$$

$(a+b)(c+d)$

FOIL

first, outside, inside, last

primero, afuera, ...

(español)

$a^3-b^3=(a-b)(a^2+ab+b^2)$

$a^3+b^3=(a+b)(a^2-ab+b^2)$

More Practice: $a^2-b^2=(a-b)(a+b)$

You will encounter problems that use both rules. You will also see some where the easiest way to proceed will be to whip out ye old algebra.

> Do you remember the formula for difference of two cubes? SOAP, anyone?

same
opposite
always
plus

expl 3: Use your algebra skills to simplify this one and then use a new rule to find $h'(x)$.

$$h(x)=\frac{(x-1)(2x^2-1)}{x^3-1}=\frac{\cancel{(x-1)}(2x^2-1)}{\cancel{(x-1)}(x^2+1x+1)}$$

$$h'(x)=\frac{(x^2+x+1)(4x)-(2x^2-1)(2x+1)}{(x^2+x+1)^2}$$

$$=\frac{\cancel{4x^3}+4x^2+4x\cancel{-4x^3}+2x-2x^2+1}{(x^2+x+1)^2}$$

$$\boxed{h'(x)=\frac{2x^2+6x+1}{(x^2+x+1)^2}}$$

expl 4: Use both rules to find this function's derivative.

$$h(x)=\frac{x+1}{x^2e^x}$$

$f(x)=x+1 \rightarrow f'(x)=1$

$g(x)=x^2e^x \rightarrow$

$g'(x)=2xe^x+x^2e^x$

$$h'(x)=\frac{x^2e^x(1)-(x+1)(2xe^x+x^2e^x)}{(x^2e^x)^2}$$

FIOL

$$=\frac{\cancel{x^2e^x}-2x^2e^x-2xe^x-x^3e^x\cancel{-x^2e^x}}{x^4\cdot e^xe^x}$$

$$=\frac{-x^3e^x-2x^2e^x-2xe^x}{x^4\cdot e^xe^x}$$

$$=\frac{\cancel{xe^x}(-x^2-2x-2)}{\cancel{x\cdot e^x}x^3e^x}=\boxed{\frac{-1(x^2+2x+2)}{x^3e^x}}=h'(x)$$