What is the rate of change (derivative) for the sine function or other trig functions?

Calculus I
Class notes
Derivatives of Trigonometry Functions (section 3.5)

Though proven in the book, I present these limits here without proof. We will use them in their own rights but they are also integral to proving the derivative formulas we will be given for trig functions.

Limits for Common Trig Functions:

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad \lim_{x\to 0}\frac{\cos x-1}{x}=0$$

expl 1: Evaluate the following limits.

This does *not* quite match the form of the limit whose value we know. Multiply top and bottom by 3 and it will. Now what?

a.) $\lim_{x\to 0}\frac{\sin 3x}{x} = \lim_{x\to 0}\frac{3\sin 3x}{3x}$

$= 3\cdot\lim_{x\to 0}\frac{\sin 3x}{3x}$

limit prop 1 (P. Dawkins)

$= 3\cdot 1 = 3$

b.) $\lim_{x\to 0}\frac{\sin 5x}{3x} = \lim_{x\to 0}\frac{\frac{5}{3}\sin 5x}{\frac{5}{3}(3x)}$

Think about what we did in part *a*. How can you use that idea?

$= \lim_{x\to 0}\frac{\frac{5}{3}\sin 5x}{5x}$

$= \frac{5}{3}\lim_{x\to 0}\frac{\sin 5x}{5x} \to 1 = \frac{5}{3}\cdot 1 = 5/3$

Algebra is often the key. Get your limit rules out too.

c.) $\lim_{x\to -3}\frac{\sin(x+3)}{x^2+8x+15} = \lim_{x\to -3}\frac{\sin(x+3)\cdot 1}{(x+3)(x+5)}$

Limit Prop 3 (P. Dawkins)

$= \lim_{x\to -3}\frac{\sin(x+3)}{(x+3)}\cdot\lim_{x\to -3}\frac{1}{x+5}$

direct substitution

$= 1\cdot\frac{1}{-3+5}$

As $x\to -3$, we see that $x+3\to 0$.

$= 1\cdot\frac{1}{2} = \frac{1}{2}$

Trig ID: $\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta - 1 = -\sin^2\theta$

$a^2 - b^2 = (a-b)(a+b)$

Can you find the two different methods?

expl 2: Evaluate the following limit.

$$\lim_{\theta\to 0}\frac{\cos^2\theta - 1}{\theta}$$

Method 1: $\lim\limits_{\theta\to 0}\dfrac{-\sin^2\theta}{\theta}$

$= \lim\limits_{\theta\to 0}\left(\dfrac{\sin\theta}{\theta}\right)(-\sin\theta)$

$= \lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}\cdot\lim\limits_{\theta\to 0}(-\sin\theta)$

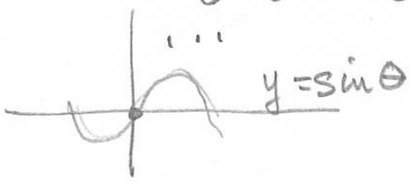
$= 1\cdot 0 = 0$

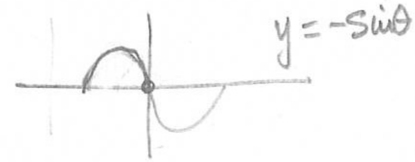
Method 2: $\lim\limits_{\theta\to 0}\dfrac{\cos^2\theta - 1^2}{\theta}$

$= \lim\limits_{\theta\to 0}\dfrac{(\cos\theta - 1)(\cos\theta + 1)}{\theta}$

$= \lim\limits_{\theta\to 0}\dfrac{\cos\theta - 1}{\theta}\cdot\lim\limits_{\theta\to 0}(\cos\theta + 1)$

...





Derivatives of Trigonometric Functions:

Again, the purpose of seeing these limits is that the book uses them to justify the following derivative rules for our six main trig functions. Gaze upon them below.

THEOREM 3.11 Derivatives of the Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x\tan x \qquad \frac{d}{dx}(\csc x) = -\csc x\cot x$$

Angles must be in **radian measure** for these formulas to work.

Review of Trig Functions:

Recall that **sin** stands for sine, **cos** stands for cosine.
Tangent (**tan**) is equal to sine / cosine.
Cotangent (**cot**) is 1 / tangent or cosine / sine.
Secant (**sec**) is 1 / cosine.
Cosecant (**csc**) is 1 / sine.

You could derive the last four rules using the derivative formulas for sine and cosine. Do you know how?

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

Quotient Rule

$$\frac{d}{dx}(fg) = f'g + fg' \qquad \frac{d}{dx}\left(\frac{f}{g}\right) = \frac{gf' - fg'}{g^2}$$

expl 3: Use these new rules along with some old favorites to find the derivatives of these functions.

a.) $y = 5x^2 + \cos x$

$$y' = 10x - \sin x$$

b.) $y = e^x(\cos x + \sin x)$

$$y' = e^x(\cos x + \sin x) + e^x(-\sin x + \cos x)$$

$$= e^x \cos x + \cancel{e^x \sin x} - \cancel{e^x \sin x} + e^x \cos x = \boxed{2e^x \cos x}$$

c.) $y = \frac{1-\sin x}{1+\sin x}$

$f(x) = 1 - \sin x$

$f'(x) = 0 - \cos x = -\cos x$

$g(x) = 1 + \sin x$

$g'(x) = \cos x$

$$y' = \frac{(1+\sin x)(-\cos x) - (1-\sin x)(\cos x)}{(1+\sin x)^2}$$

$$= \frac{-\cos x - \cancel{\sin x \cos x} - \cos x + \cancel{\sin x \cos x}}{(1+\sin x)^2}$$

MML:

$$\boxed{y' = \frac{-2\cos x}{(1+\sin x)^2}} \quad \text{OR} \quad \boxed{y' = \frac{-2\cos x}{1 + 2\sin x + \sin^2 x}}$$

Higher Derivatives:

expl 4: Find y'' for $y = \cot\theta$.

Later, the Chain Rule will give us an alternative. For now, we'll employ the Product Rule.

$$y' = -\csc^2\theta$$

$$y' = (-\csc\theta)(\csc\theta)$$

$$y'' = (+\csc\theta \cdot \cot\theta)\csc\theta + (-\csc\theta)(-\csc\theta\cot\theta)$$

$$y'' = \csc^2\theta \cot\theta + \csc^2\theta\cot\theta$$

$(a \cdot b) \cdot a$
$= a \cdot (b \cdot a)$
$= a(ab)$
$= (aa)b = a^2 b$

$$\boxed{y'' = 2\csc^2\theta\cot\theta}$$