2:00

Recall that the derivative is the slope of the tangent line at a certain point. This slope tells us how quickly the function is changing.

Calculus I
Class notes
Derivatives as Rates of Change (section 3.6)

We have already seen how if $s(t)$ is the position function of an object in motion, then its derivative $s'(t)$ tells us the object's velocity, or rate of change of its position.

We have seen that the units for this rate of change are based on the fact that this is no more than slope (difference of $y$-values divided by difference of $x$-values). Hence if $s(t)$ is in feet and $t$ is in seconds, then $s'(t)$ would be in feet per second.

We will start to use the function name $v(t) = s'(t)$ to mean the object's **velocity**. What is the rate of change of this velocity function? In other words, what would we call $v'(t) = s''(t)$? What would the units be for such a function?

acceleration

$a(t) = v'(t) = s''(t)$

The units would be $\frac{ft/sec}{sec}$

or $ft/sec^2$

Don't forget about the alternative notation $\frac{ds}{dt}$ and $\frac{d^2s}{dt^2}$.

$s'$ or $v$ $\quad$ $s''$ or $a$

We will cover various functions and their rates of change (derivatives). A commonly explored one is that $s(t)$, $v(t)$, and $a(t)$ we started with. We will also play with cost, profit, and revenue and their derivatives ***marginal*** **cost, profit, and revenue**. Since we're down that road, we'll also learn about ***average*** **cost, profit, and revenue**.

(has nothing to do with derivatives.

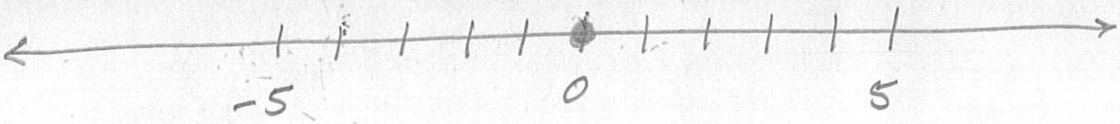
~~**Definition: Displacement:** how far an object has travelled. The **displacement** of an object~~ between $t = a$ and $t = a + \Delta t$ is $\underbrace{\cancel{s(t)}}_{\Delta s} = s(a + \Delta t) - s(a)$. We call this $\Delta t$ the **elapsed time**.

This is pronounced "delta $t$".

**Definition: Speed: Speed** is the magnitude of velocity. Algebraically, we say that **speed** = $|v|$.

Velocity is a **vector** quantity. It has magnitude *and* direction. Speed is **scalar** and only describes magnitude.

An object's velocity might be −70 ft/sec. The negative sign indicates it's going down. Its speed is said to be 70 ft/sec (removing the direction).

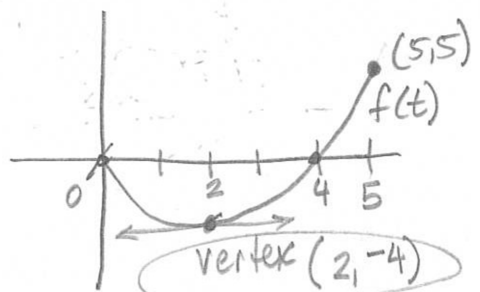



**Horizontal Motion:** $s(t)$ or $f(t)$

expl 1: Suppose $s = f(t) = t^2 - 4t$, $0 \le t \le 5$, gives the position of an object moving horizontally along a line after $t$ seconds. Here, $s$ is measured in inches, with $s > 0$ corresponding to positions right of the origin.

Imagine it on your table moving along a number line.

a.) Graph the position function $f(t)$ for the given interval.




$f(2) = 2^2 - 4(2)$
$= -4$

$f(5) = 5^2 - 4 \cdot 5$
$= 5$

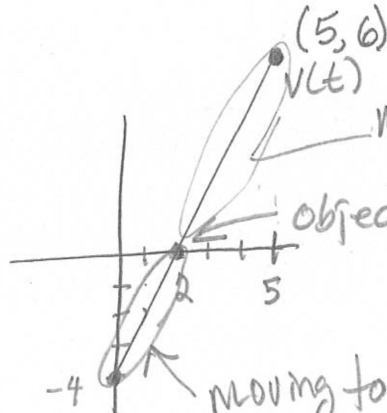
The parabola's vertex is at $\left(\frac{-b}{2a}, s\left(\frac{-b}{2a}\right)\right)$.

$[0,5] \times [-10,10]$

b.) Analyze the graph above to find when the object is stationary, moving to the left, and moving to the right? It starts at $s=0$ $(t=0)$ and moves to left, getting to $s=-4$ at $t=2$. It stops temporarily, then moves to right getting to $s=5$ at $t=5$.

c.) Find and graph the velocity function. Label the parts of this graph that correspond to the object being stationary, moving to the left, and moving to the right.

$v(t) = s'(t) = 2t - 4$




moving to right on interval $(2,5]$

object is stopped at $t=2$.

moving to left on the interval $[0,2)$

d.) Determine the object's velocity and acceleration when $t = 1$ second.

$v(1) = 2 \cdot 1 - 4$

$v(1) = -2$ in/sec

The object is moving at $-2$ in/sec when $t=1$ second.

$a(t) = v'(t)$
$= 2$ in/sec$^2$

$a(1) = 2$ in/sec$^2$

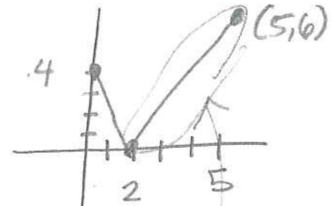
The vel. changes at the rate of 2 in/sec every second.

e.) Determine the object's acceleration when its velocity equals 0 inches per second.

Accel is 2 in/sec$^2$ for all $t$.

So, $a(t) = 2$ when $v(t) = 0$ in/sec.

Consider speed graph




expl 1 (continued):

f.) Find the object's **speed** for all integer values in the interval [0, 5]. On what interval(s) is the object's speed increasing?

| $t$ | Speed $= \lvert v \rvert = \lvert 2t-4 \rvert$ |
|---|---|
| 0 | $\lvert -4 \rvert = 4$ |
| 1 | $\lvert -2 \rvert = 2$ |
| 2 | $\lvert 0 \rvert = 0$ |
| 3 | $\lvert 2 \rvert = 2$ |
| 4 | $\lvert 4 \rvert = 4$ |
| 5 | $\lvert 6 \rvert = 6$ |

The speed increases on the interval $(2, 5]$.

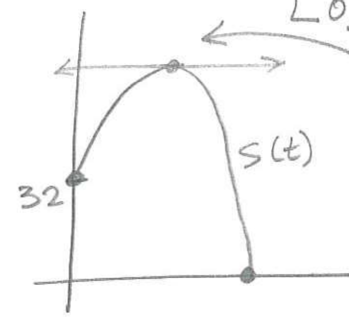
Seen on graph

**Motion in Gravitational Fields:**

expl 2: We throw a stone upward off a cliff edge and let it fall to the beach below. (No people were harmed during the making of this example.) Its position (height, in feet) is given by $s(t) = -16t^2 + 64t + 32$ where $t$ is time in seconds.

$[0, 10] \times [0, 100]$

a.) Draw a rough graph of $s(t)$.




max height $v(t) = 0$

b.) Find the stone's velocity function $v(t)$.

$v(t) = s'(t) = -32t + 64$

c.) When does the stone reach its highest point? What is the maximum height it achieves?

$v(t) = 0 = -32t + 64$

$-64 = -32t$

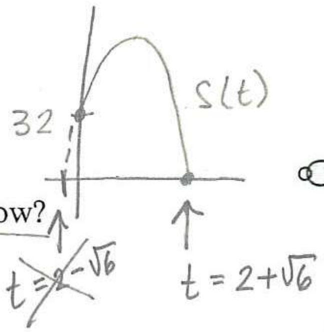
$t = 2$ sec (When stone hits max height)

$s(2) = -16 \cdot 2^2 + 64 \cdot 2 + 32$

$= 96$ feet  max height

At the highest point, the velocity changes from positive to negative, with 0 at the peak.

$\underline{\underline{a}}t^2 + \underline{\underline{b}}t + \underline{\underline{c}} = 0$




Get out your quadratic formula to solve $s(t) = 0$.

expl 2 continued:

d.) When does the stone hit the beach below?

$s(t) = 0$

$-16t^2 + 64t + 32 = 0$

$-16(t^2 - 4t - 2) = 0$

$t = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{4 \pm \sqrt{(-4)^2 - 4(1)(-2)}}{2(1)}$

$= \dfrac{4 \pm \sqrt{24}}{2}$

$t = \dfrac{4 \pm 2\sqrt{6}}{2}$

$t = \dfrac{4}{2} \pm \dfrac{2\sqrt{6}}{2}$

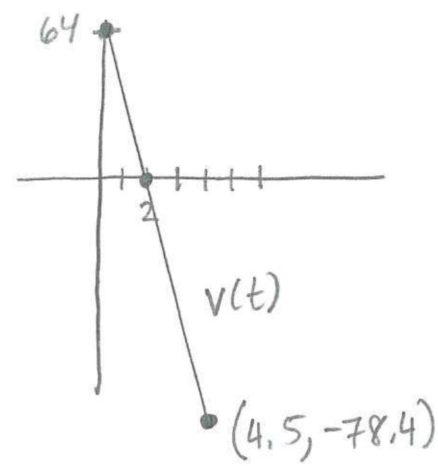
$t = 2 \pm \sqrt{6}$

(Disregard neg. value)

$t = 2 + \sqrt{6}$ seconds

(about 4.5 sec.)

Graph $v(t)$:




e.) With what velocity does the stone hit the beach?

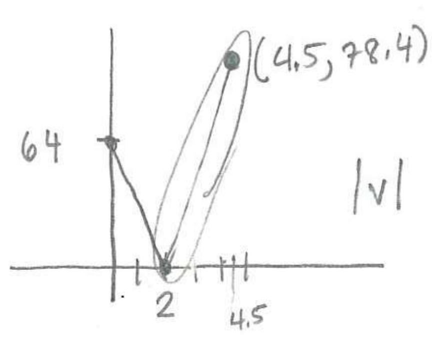
$v(t) = -32t + 64$

$v(2+\sqrt{6}) = -32(2+\sqrt{6}) + 64$

$\approx -78.4$ ft/sec

f) On what interval(s) is speed increasing?

speed = $|v|$




Speed increases on interval $[2, 2+\sqrt{6}]$.

**Rates of Change, in General:**

We are working with specific examples here but understand that these ideas will work for any function. Let's say you know a function $p(t)$ measures something like a population *or* price per item *or* number of internet users, etc. Here, $t$ is taken to be time where $t \geq 0$.

This function's **instantaneous rate of change** is its derivative $p'(t)$ or $\frac{dp}{dt}$. This measures how fast $p(t)$ is changing with respect to time.

The units of this rate of change will always be "units of $p(t)$ divided by units of $t$".

Marginal and **Average Cost, Profit, and Revenue:**

Let $C(x)$ be the cost of producing $x$ items. Perhaps we will be given the function $C(x) = -0.02x^2 + 50x + 100$ dollars.

Recall that profit equals revenue minus cost.

The **Average Cost per Item** would be given by $\overline{C}(x) = \frac{C(x)}{x}$.

Pronounced "C bar".

This should make sense if you picture a total cost of \$500 for 100 items.

The **Marginal Cost** is the approximate cost to produce *one additional item* after producing $x$ items. And that sounds like rate of change. Hence, marginal cost is $C'(x)$.

Further, let $p(x)$ denote the sale price per item if $x$ items are sold.

Notice this is *lowercase p*.

We would find **Revenue** (which is total money brought in) as the product $R(x) = x \cdot p(x)$.

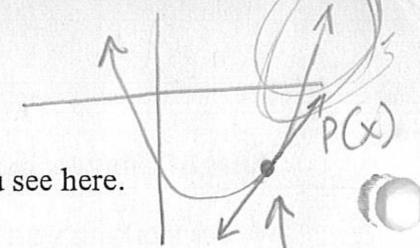
Which leaves us **Profit** (the money we really made) as

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= x \cdot p(x) - C(x) \end{aligned}$$

Here's the *uppercase P*.

We are in the US so our units for cost, revenue, and profit will usually be in dollars with the units of their derivatives derived accordingly.

It is *not* a long stretch to define average or marginal revenue and profit as you see here.




The **Average Revenue per Item** would be given by $\bar{R}(x) = \dfrac{R(x)}{x}$. The **Marginal Revenue,** $R'(x)$, is the approximate revenue gained by selling *one additional item* after selling $x$ items.

slope has units $\dfrac{\text{dollars of profit}}{\text{number of items}}$

The **Average Profit per Item** would be given by $\bar{P}(x) = \dfrac{P(x)}{x}$. The **Marginal Profit,** $P'(x)$, is the approximate profit gained by selling *one additional item* after selling $x$ items.

expl 3: A company has a cost function of $C(x) = -0.02x^2 + 50x + 100$ dollars when they produce $x$ items. The sale price per item is given as $p(x) = 100$ dollars.

a.) Find the profit function $P(x)$.

$$P(x) = R(x) - C(x) = x \cdot p(x) - C(x)$$
$$= 100x - (-0.02x^2 + 50x + 100)$$
$$= 100x + 0.02x^2 - 50x - 100$$
$$P(x) = 0.02x^2 + 50x - 100 \text{ (dollars)}$$

b.) Find the average profit and marginal profit functions.

avg profit

$$\bar{P}(x) = P(x)/x$$
$$= \frac{0.02x^2 + 50x - 100}{x}$$
$$\bar{P}(x) = 0.02x + 50 - 100/x$$

Marginal profit

$$P'(x) = 0.04x + 50$$

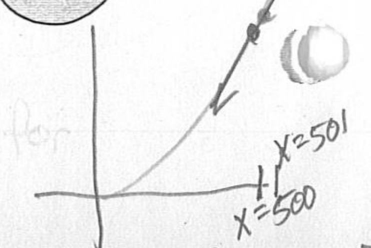
c.) Find the average profit and marginal profit when $x = 500$ items are sold. Interpret.

$$\bar{P}(500) = 0.02(500) + 50 - 100/500$$
$$= \$59.80/\text{item}$$

If we make & sell 500 items, we'll make an average profit of \$59.80 per item.

The marginal profit is the profit we make on the 501$^{st}$ item.

slope of tangent line = \$70




[0,600] x [0,50,000]

$$P'(500) = 0.04(500) + 50$$
$$= \$70$$ → So, we'll make a profit of \$70 on the 501$^{st}$ item.