12:00

Many functions are the composition of two functions whose derivative rules we know.

Calculus I
Class notes
The Chain Rule (section 3.7)

What's the derivative of the function $y = e^{-3x}$?

We know that $\frac{d}{dx}\left(e^x\right) = e^x$ but we *cannot* simply say that $y' = e^{-3x}$ because the exponent is *not* just x. In fact, this exponent of $-3x$ can be thought of as a function unto itself. And, that's the key. We need to start seeing some functions as the composition of two functions whose derivative rules we know.

We can think of $y = e^{-3x}$ as $f(g(x))$ where $g(x) = -3x$ and $f(u) = e^u$. Find the formula for the composed function $f(g(x))$ to verify that.

$f(g(x)) = f(-3x)$
$= e^{-3x}$ ✓

You can review "function composition" by searching for it on www.khanacademy.org.

For this and other complicated functions, we will use the Chain Rule. Often, algebra along with our previous rules can be used like in the case of $\frac{d}{dx}\left((2x+6)^3\right)$. How would you do that? $(2x+6)^3 = (2x+6)(2x+6)(2x+6)$ ... Get it in the form of many terms and...

However, once learned, the Chain Rule will prove easier. By the way, do you see the functions f and g that make up the function $h(x) = (2x+6)^3$?

$f(u) = u^3$; $g(x) = 2x+6$ such that $f(g(x)) = (2x+6)^3$

Check
$f(g(x))$
$= f(2x+6)$
$= (2x+6)^3$ ✓

THEOREM 3.12 The Chain Rule
Suppose $y = f(u)$ is differentiable at $u = g(x)$ and $u = g(x)$ is differentiable at x. The composite function $y = f(g(x))$ is differentiable at x, and its derivative can be expressed in two equivalent ways.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad (1)$$

OR

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x) \qquad (2)$$

The derivative of f evaluated at g multiplied by the derivative of g evaluated at x.

Once you get the hang of it, this will come naturally. The book has this handy set of steps to help the beginner.

PROCEDURE Using the Chain Rule

Assume the differentiable function $y = f(g(x))$ is given.

1. Identify an outer function f and an inner function g, and let $u = g(x)$.
2. Replace $g(x)$ with u to express y in terms of u:

$$y = f(\underbrace{g(x)}_{u}) \Rightarrow y = f(u).$$

3. Calculate the product $\frac{dy}{du} \cdot \frac{du}{dx}$.
4. Replace u with $g(x)$ in $\frac{dy}{du}$ to obtain $\frac{dy}{dx}$.

It does feel cumbersome at first but you will learn to do it with more fluidity.

expl 1a: Find y' for $y = e^{-3x}$.

① $f(u) = e^u$; $g(x) = -3x = u$

$\rightarrow y = f(g(x)) = f(-3x) = e^{-3x}$

② $y = e^{-3x}$

$y = e^u$

③ $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= e^u \cdot (-3)$

$= -3e^u$

④ $= -3e^{-3x}$

So, y' or $\frac{dy}{dx} = -3e^{-3x}$

3.3 Power Rule $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$

expl 1b: Find y' for ~~$y = 5e^{-3x}$~~. $y = \sqrt{-3x}$

$y = (-3x)^{1/2}$

(Chain Rule)

$y' = f'(g(x)) \cdot g'(x)$

$= \frac{1}{2}(-3x)^{-1/2} \cdot (-3)$

$\boxed{y' = -\frac{3}{2}(-3x)^{-1/2}}$

$\frac{d}{dx}(f(g(x)) = f'(g(x)) \cdot g'(x)$ from page 1

$f(u) = u^{1/2}$

$g(x) = -3x$

such that $f(g(x)) = (-3x)^{1/2}$

$g'(x) = -3$

$f'(u) = \frac{1}{2}u^{-1/2}$

$f'(g(x)) = \frac{1}{2}(-3x)^{-1/2}$

Chain Rule for Powers:
Since we do this a lot, it is somewhat helpful that the book draws out this specifically for its own formula. In fact, it is just the Chain Rule combined with the Power Rule from earlier.

> **THEOREM 3.13 Chain Rule for Powers**
> If g is differentiable for all x in its domain and p is a real number, then
> $$\frac{d}{dx}\left((g(x))^p\right) = \underbrace{p(g(x))^{p-1}}_{\text{Power Rule}} g'(x).$$

$f(u) = u^p$ ← from Chain Rule

expl 2: Find $\frac{d}{dx}\left((2x+6)^3\right) = 3(2x+6)^2 \cdot 2$
$= 6(2x+6)^2$

$g(x) = 2x+6$
$p = 3$

expl 3: Find $\frac{d}{dx}\left(\sqrt{x^2+1}\right)$. $= \frac{d}{dx}\left((x^2+1)^{1/2}\right)$

Do you recall the exponent equivalency for roots?

$= \frac{1}{2}(x^2+1)^{-1/2} \cdot (2x)$
$= x(x^2+1)^{-1/2}$

The Orig. Chain Rule
$f(u) = u^{1/2} \to f'(u) = \frac{1}{2}u^{-1/2}$
$g(x) = x^2+1 \to g'(x) = 2x$
$\frac{d}{dx}(x^2+1)^{1/2} = f'(g(x)) \cdot g'(x)$
$= \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x$
$= x(x^2+1)^{-1/2}$

$\frac{d}{dx}(fg) = f'g + fg'$

Product Rule, anyone?

expl 4: Find y' for $y = x(x+2)^{1/3}$.

✓ Product Rule

$y' = 1(x+2)^{1/3} + x \cdot \frac{1}{3}(x+2)^{-2/3}$

$y' = (x+2)^{1/3} + \frac{x}{3}(x+2)^{-2/3}$

OR

$y' = (x+2)^{1/3} + \frac{x}{3(x+2)^{2/3}}$

$g(x) = (x+2)^{1/3}$
$g'(x) = \frac{1}{3}(x+2)^{-2/3} \cdot 1$ Chain Rule

Question: Can I do this $x(x+2)^{1/3} = (x^2+2x)^{1/3}$?
No order of operations

$f(u) = u^{1/3}$
$g(x) = x^2+2x$

$$f(g(x)) = \tan(5x^2)$$

expl 5: Find y' for $y = \tan\left(5x^2\right)$.

$$f(u) = \tan u \qquad g(x) = 5x^2$$

$$f'(u) = \sec^2 u \qquad g'(x) = 10x$$

Chain Rule

$$\frac{d}{dx}\left(\tan(5x^2)\right) = f'(g(x)) \cdot g'(x)$$

$$= \sec^2(5x^2) \cdot 10x$$

$$\frac{d}{dx}(5x^2) = 10x$$

$$y' = 10x\sec^2(5x^2)$$

expl 6: Find y' for $y = \sin^5\left(\cos(3x)\right)$.

This is also written as $y = \left(\sin\left(\cos(3x)\right)\right)^5$.

$$y = (\sin(\cos(3x)))^5$$

Power Rule

$$y' = 5(\sin(\cos(3x)))^4 \cdot \underbrace{\frac{d}{dx}(\sin(\cos(3x)))}_{\cos(\cos(3x)) \cdot \frac{d}{dx}(\cos(3x))}$$

comes from Chain Rule

$$\cos(\cos(3x)) \cdot \underbrace{\frac{d}{dx}(\cos(3x))}_{-\sin(3x) \cdot \frac{d}{dx}(3x)}$$

Chain Rule

~~*Two*~~ three instances of the Chain Rule?!?

$$-\sin(3x) \cdot \frac{d}{dx}(3x)$$

Chain Rule

$$y' = 5(\sin(\cos(3x)))^4 \cdot \cos(\cos(3x))(-\sin(3x)) \cdot 3$$

$$\boxed{y' = -15(\sin(\cos(3x)))^4 \cdot \cos(\cos(3x)) \cdot \sin(3x)}$$

Remember
$$\cos(\cos(3x)) \neq \cos^2(3x)$$

2:00

$\sin(\pi t/12) = \sin\left(\frac{\pi}{12} \cdot t\right)$

expl 7: The total energy in megawatt-hours (MWh) used by a town is given by

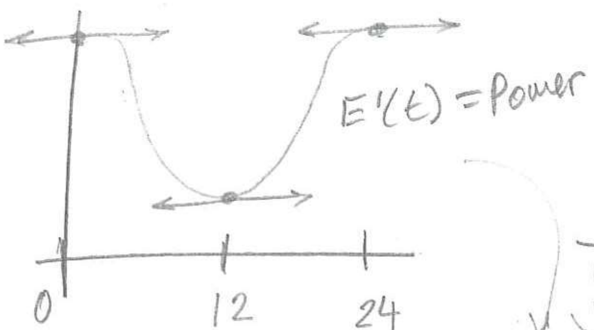
$E(t) = 400t + \dfrac{2400}{\pi}\sin\left(\pi t/12\right)$, $t \geq 0$ where t is measured in hours with $t = 0$ corresponding to noon.

a.) Find the power, or the *rate* of energy consumption, $P(t) = E'(t)$, in terms of megawatts. MW

$P(t) = E'(t) = 400 + \frac{2400}{\pi} \cos(\pi t/12) \cdot \pi/12$ Chain Rule

$E'(t) = 400 + 200\cos(\pi t/12)$ MW

b.) Graph $P(t) = E'(t)$ in the window [0, 24] x [0, 700].



Notice the slopes of tangent lines at the maxes and min are 0.

To find maxes and mins on the graph of $E'(t)$, we'll derive $E''(t)$ and set is equal to 0.

c.) Algebraically find the time of day that the power is at a maximum? What is the power at that time?

We need $E''(t)$. Do you see why?

$E'(t) = 400 + 200\cos(\pi t/12)$ Chain Rule

$E''(t) = 0 - 200\sin(\pi t/12) \cdot \pi/12$

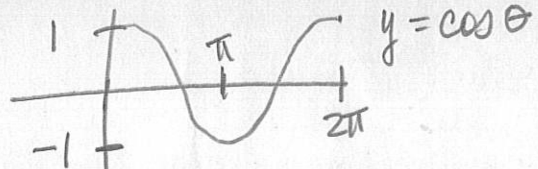
$E''(t) = \dfrac{-200\pi}{12}\sin(\pi t/12)$

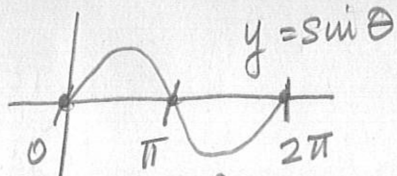
To find maxes, mins, set $E''(t) = 0$

$0 = \dfrac{-200\pi}{12}\sin(\pi t/12)$

Continued discussion of part *c* is on the next page.

$0 = \sin(\pi t/12)$





expl 7c continued:

$\sin\theta = 0$ for $\theta = 0, \pi, 2\pi, \ldots$

So far, we should be at $\sin\left(\frac{\pi t}{12}\right) = 0$. Do you recall the graph of $y = \sin(\theta)$ for $0 \le \theta \le 2\pi$?

Draw it here to help find the t-values that make $\sin\left(\frac{\pi t}{12}\right) = 0$.

$\frac{\pi t}{12} = 0$ or $\frac{\pi t}{12} = \pi$ or $\frac{\pi t}{12} = 2\pi$

$t = 0$ $\quad t = 12\pi/\pi \quad t = 24\pi/\pi$

$t = 12 \quad t = 24$

So, $E'(0) = 400 + 200\cos\left(\pi \cdot 0/12\right)$

$= 400 + 200\underbrace{\cos(0)}_{1} = 600$ MW

This is noon.

This is the maximum.

$E'(12) = 400 + 200\cos\left(\pi \cdot 12/12\right)$

$= 400 + 200\underbrace{\cos(\pi)}_{-1}$

$= 400 - 200 = \boxed{200 \text{ MW}}$

This is midnight.

d.) Your work also revealed a minimum power. When does that occur and at what time will the city see that?

Minimum power happens at midnight ($t = 12$) and it's 200 MW.