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Often, functions we are given have $y$ isolated. But what if they don't?

Calculus I
Class notes
Implicit Differentiation (section 3.8)

We have been finding the derivatives of functions that are in the form $y = blah\ blah\ blah$ which is said to be an **explicitly** defined function. But there are lots of functions that are **implicitly** defined such as $3x+4y^3=7$ whose derivative we may need. What's a student to do?

Coming to our rescue, we have **implicit differentiation**. We will find the derivatives of all terms as our rules govern, keeping in mind that the derivative of $y$ with respect to $x$ is written as $\frac{dy}{dx}$.

Inherent in what we do here is the Chain Rule. It is nice to *not* lose sight of that as we go.

A nasty, quarrelsome student might ask, "Can't we just solve for $y$ and then differentiate?" To that I answer, "Ah, well, yes, … but sometimes that's *not* possible or even harder than implicitly differentiating."

Let's jump right in with examples. We will go term by term. Again, recall that the derivative of $y$ with respect to $x$ is written as $\frac{dy}{dx}$. Always isolate $\frac{dy}{dx}$ to finish.

For the second term, we use the Chain Rule because $y$ is a function in $x$.

expl 1: Calculate $\frac{dy}{dx}$.

$3x+4y^3=7$

Chain Rule

$$3 + 12y^2\frac{dy}{dx} = 0$$

$$12y^2\frac{dy}{dx} = -3$$

$$\frac{dy}{dx} = \frac{-3}{12y^2}$$

$$\boxed{\frac{dy}{dx} = \frac{-1}{4y^2}}$$

Sometimes it will *not* be possible to solve for $y$ for traditional differentiation.

We will see a trick where we combine $\frac{dy}{dx}$ terms with the distribution property.

expl 2: Calculate $\frac{dy}{dx}$.

$\sin x + \sin y = y$

$$\cos x + \cos y \cdot \frac{dy}{dx} = \frac{dy}{dx}$$

$$\cos x = \frac{dy}{dx} - \cos y \frac{dy}{dx}$$

$$\cos x = \frac{dy}{dx}(1 - \cos y)$$

$$\frac{dy}{dx} = \frac{\cos x}{1 - \cos y}$$

$$\frac{d}{dx}(fg) = f'g + fg'$$

To find the derivative of $e^{xy}$, we will need the Chain Rule (because the exponent is *not* just $x$). But how do we find the derivative of $xy$?

expl 3: Calculate $\frac{dy}{dx}$.

$e^{xy} = 2y$

$$\frac{d}{dx}(e^{xy}) = \frac{d}{dx}(2y)$$

$$e^{xy}\left(y + x\frac{dy}{dx}\right) = 2\frac{dy}{dx}$$

$$ye^{xy} + xe^{xy}\frac{dy}{dx} = 2\frac{dy}{dx}$$

$$ye^{xy} = 2\frac{dy}{dx} - xe^{xy}\frac{dy}{dx}$$

$$ye^{xy} = \frac{dy}{dx}(2 - xe^{xy})$$

$$\boxed{\frac{dy}{dx} = \frac{ye^{xy}}{2 - xe^{xy}}}$$

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$$

We want $\frac{d}{dx}(e^{xy})$ where $f(u) = e^u$

$g(x) = xy$

So $\frac{d}{dx}(e^{xy}) = f'(g(x)) \cdot g'(x)$

$= e^{xy}\left(1y + x\,dy/dx\right)$

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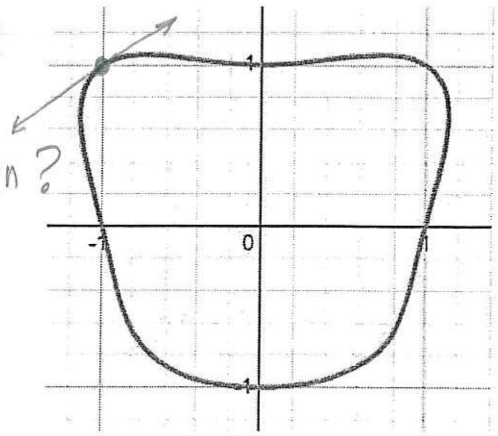
expl 4: For the following implicitly defined relationship, determine the equation of the line tangent to the curve at the point $(-1, 1)$.

$x^4 - x^2 y + y^4 = 1$

$\frac{dy}{dx}$ = slope of the tang. line

Here's a graph I made with the help of the good people at www.desmos.com. Find the point $(-1, 1)$ and draw in the tangent line.

eqn ?




How would you algebraically show that this point is on the curve?

$\frac{d}{dx}(fg) = f'g + fg'$

First, find $\frac{dy}{dx}$ because that's the slope of the tangent line for any point $(x, y)$ we are given.

Then what?

$$\frac{d}{dx}\left(x^4 - x^2 y + y^4\right) = \frac{d}{dx}(1)$$

Product Rule / Chain Rule

$$4x^3 - \left(2xy + x^2 \frac{dy}{dx}\right) + 4y^3 \cdot \frac{dy}{dx} = 0$$

$$4x^3 - 2xy - x^2 \frac{dy}{dx} + 4y^3 \frac{dy}{dx} = 0$$

$$4x^3 - 2xy = x^2 \frac{dy}{dx} - 4y^3 \frac{dy}{dx}$$

$$4x^3 - 2xy = \frac{dy}{dx}\left(x^2 - 4y^3\right)$$

$$\frac{dy}{dx} = \frac{4x^3 - 2xy}{x^2 - 4y^3}$$

So, slope of tang. line at $(-1, 1)$ is

$$\left.\frac{dy}{dx}\right|_{(-1,1)} = \frac{4(-1)^3 - 2(-1)(1)}{(-1)^2 - 4(1)^3} = \frac{-4+2}{1-4} = \frac{-2}{-3} = \frac{2}{3}$$

From 3.1 – the eqn of the tangent line at $(-1, 1)$ is

$$y - f(a) = m_{tan}(x - a)$$

$$y - 1 = \frac{2}{3}(x - {-1})$$

$$y - 1 = \frac{2}{3}x + \frac{2}{3}$$

$$\boxed{y = \frac{2}{3}x + \frac{5}{3}}$$

Vertical lines have undefined slopes. (zero on denom. of fraction)
Horizontal lines have 0 slopes. (zero on numerator of fraction)

expl 5: Consider the relationship $x + y^3 - y = 1$. Determine which points on this curve have a vertical or horizontal tangent line.

$\frac{dy}{dx}$ = slope of tang. lines

First, find $\frac{dy}{dx}$ because that's the slope of the tangent line for any point $(x, y)$ we are given.

Now, what does it mean for this tangent line to be vertical or horizontal, as far as slope is concerned? Give answers as ordered pairs in exact form.

You will end up in radical form. Do *not* round.

$$\frac{d}{dx}(x+y^3-y) = \frac{d}{dx}(1)$$

$$1 + 3y^2\frac{dy}{dx} - 1\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(3y^2-1) = -1$$

$$\frac{dy}{dx} = \frac{-1}{3y^2-1} \quad \text{OR} \quad \frac{dy}{dx} = \frac{1}{1-3y^2}$$

Horizontal Tangent Lines occur where the top is 0.
This never happens → So, there are no hor. tangent lines.

Vertical Tangent Lines occur where bottom is 0.

Solve $3y^2-1=0$

$$3y^2 = 1$$

$$y^2 = 1/3$$

$$\sqrt{y^2} = \sqrt{1/3}$$

$$\boxed{y = \pm\left(\tfrac{1}{3}\right)^{1/2}}$$

Find x for when $y = (1/3)^{1/2}$

$$x = 1 - y^3 + y$$

$$= 1 - \left(\left(\tfrac{1}{3}\right)^{1/2}\right)^3 + \left(\tfrac{1}{3}\right)^{1/2}$$

$$= 1 - \left(\tfrac{1}{3}\right)^{1/2}\left[\left(\left(\tfrac{1}{3}\right)^{1/2}\right)^2 - 1\right]$$

$$= 1 - \left(\tfrac{1}{3}\right)^{1/2}\left(-\tfrac{2}{3}\right)$$

$$x = 1 + \tfrac{2}{3}\left(\tfrac{1}{3}\right)^{1/2}$$

Find x for when $y = -(1/3)^{1/2}$

$$x = 1 - y^3 + y$$

$$= 1 - \left(-(1/3)^{1/2}\right)^3 - (1/3)^{1/2}$$

$$= 1 + \left((1/3)^{1/2}\right)^3 - (1/3)^{1/2}$$

$$= 1 + \left(\tfrac{1}{3}\right)^{1/2}\left[\left(\left(\tfrac{1}{3}\right)^{1/2}\right)^2 - 1\right]$$

$$=$$

$$x = 1 + (1/3)^{1/2}(-2/3)$$

$$\boxed{x = 1 - \tfrac{2}{3}\left(\tfrac{1}{3}\right)^{1/2}} \quad \text{OR}$$

$$\boxed{x = 1 - \tfrac{2}{3}\sqrt{\tfrac{1}{3}}}$$

One point where vert tang line exists: $\left(1 - \frac{2}{3}\left(\frac{1}{3}\right)^{1/2}, -(1/3)^{1/2}\right)$

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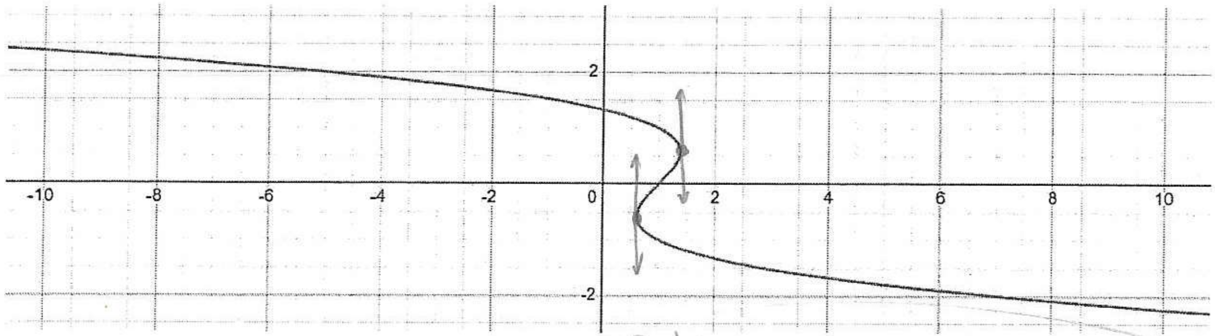
So, vert tangent lines should exist at the points $\left(1-\frac{2}{3}\left(\frac{1}{3}\right)^{1/2}, -\left(1/3\right)^{1/2}\right) \approx (0.6, -0.6)$ and $\left(1+\frac{2}{3}\left(\frac{1}{3}\right)^{1/2}, \left(\frac{1}{3}\right)^{1/2}\right) \approx (1.4, 0.6)$

expl 5 exploration: Here we see the graph of $x + y^3 - y = 1$. (www.desmos.com) Estimate your points on the graph.




Chain Rule

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x)$$

**Higher-Order Implicit Differentiation:**

Use the procedure to find and isolate $\frac{dy}{dx}$. Then take the derivative of that and you'll have $\frac{d^2y}{dx^2}$.

expl 6: Find $\frac{d^2y}{dx^2}$ of $2x^2 + y^2 = 4$.

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$$

$$\frac{d}{dx}(2x^2 + y^2) = \frac{d}{dx}(4)$$

$$4x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-4x}{2y} \rightarrow \frac{dy}{dx} = \frac{-2x}{y}$$

$$\frac{d^2y}{dx^2} = \frac{-2y + 2x(dy/dx)}{y^2}$$

$$= \frac{-2y + 2x(-2x/y)}{y^2}$$

$$\frac{d^2y}{dx^2} = \frac{-2y - 4x^2/y}{y^2}$$

May need to simplify:

$$\frac{d^2y}{dx^2} = \frac{\frac{-2y}{1} - \frac{4x^2}{y}}{y^2}$$

$$= \frac{\frac{-2y^2}{y} - \frac{4x^2}{y}}{y^2}$$

$$= \frac{\frac{-2y^2 - 4x^2}{y}}{\frac{y^2}{1}}$$

$$= \frac{-2y^2 - 4x^2}{y} \cdot \frac{1}{y^2}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{-2y^2 - 4x^2}{y^3}}$$