Two or more variables will be related to one another by their rates of change.

Calculus I
Class notes
Related Rates (section 3.9)

Here, we will see two or more variables. One variable will change with respect to time. The other variables will change with respect to the first variable (and therefore, with respect to time as well). The rates of change of the variables are *related* to one another.

Here's an example.

$\frac{dA}{dt} = ?$

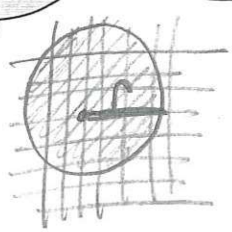
expl 1: A restaurant supplier services the restaurants in a circular area in such a way that the radius r is increasing at the rate of 2 miles per year at the moment that the radius of service is 5 miles. How fast is the service area increasing?

What are we asked to find? Evaluate it when $r = 5$. Include units.

Draw a city map with a circle representing the radius of the service area. What is the area of this circle?

Area (A) is dependent on **Radius (r)** which changes with respect to **Time (t)**.

We are given $\frac{dr}{dt}$. What is it?



$$A = \pi r^2$$

$$\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2)$$

$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$

$$\left.\frac{dr}{dt}\right|_{r=5} = 2 \text{ mi/year}$$

$$\left.\frac{dA}{dt}\right|_{r=5} = 2\pi \cdot 5 \cdot 2$$

$$= 20\pi$$

$$\approx 62.8 \text{ square miles / year}$$

At the point at which the radius is 5 miles, the service area is increasing at the rate of 62.8 sq mi/year.

Power Rule: $\frac{d}{dt}(t^n) = nt^{n-1}$

Chain Rule: $\frac{d}{dt}(f(g)) = f'(g)\cdot g'$

Product Rule: $\frac{d}{dt}(fg) = f'g + fg'$

expl 2: Assume $w = x^2y^4$ where x and y are functions in t. Find $\frac{dw}{dt}$ when $x = 3$, $\frac{dx}{dt} = 2$, $\frac{dy}{dt} = 4$, and $y = 1$.

What rules do we need to find the derivative of w?

$$\frac{d}{dt}(w) = \frac{d}{dt}(x^2y^4)$$

$$\frac{dw}{dt} = 2x\cdot\frac{dx}{dt}y^4 + x^2\cdot 4y^3\frac{dy}{dt}$$

$$\left.\frac{dw}{dt}\right|_{x=3,\ dx/dt=2,\ dy/dt=4,\ y=1} = 2\cdot 3\cdot 2\cdot 1^4 + 3^2\cdot 4\cdot 1^3\cdot 4 = 156$$

expl 3: A spherical snowball melts at a rate proportional to its surface area. Show that the rate of change of the radius is constant.

The surface area of a sphere with radius r is $SA = 4\pi r^2$.

Its volume is $V = \frac{4}{3}\pi r^3$.

Recall: If y is **directly proportional** to x, then $y = k\cdot x$ for some $k \in \mathbb{R}$.

$\frac{dV}{dt} = k\cdot SA$ for some real number k.

$$\cancel{\tfrac{4}{3}}\cancel{\pi}\cancel{3}\cancel{r^2}\frac{dr}{dt} = k\cdot\cancel{4}\cancel{\pi}\cancel{r^2}$$

$\frac{dr}{dt} = k$ which was defined to be a real number. Hence, the rate of the change of the radius is constant.

QED.

2:00

$\frac{dx}{dt} = 5$ ft/sec $\quad \left.\frac{dl}{dt}\right|_{l=120} = ?$

expl 4: Once Kate's kite reaches a height above her hand of 50 feet, it goes *no* higher but drifts to the east at 5 feet/second. How fast is the string running through her hand at the moment when she has released 120 feet of string? Include units.

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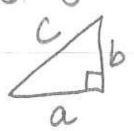
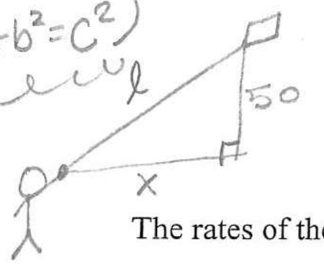


$a^2+b^2=c^2$

Let l be the length of her string. Let x be the horizontal distance from Kate to her kite. Both are functions in t.

Draw a picture. Label the right triangle formed by the string and the given height of 50 feet.



The rates of these variables are $\frac{dl}{dt}$ and $\frac{dx}{dt}$. Which have they given us and which do we need to find?

dx/dt

Form an equation from your right triangle that shows how l and x are related. Then find the derivative of each side with respect to t. Use implicit differentiation and the Chain Rule.

$$x^2 + 50^2 = l^2$$
$$x^2 + 2500 = l^2$$
$$\frac{d}{dt}(x^2+2500) = \frac{d}{dt}(l^2)$$
$$2x\frac{dx}{dt} + 0 = 2l\frac{dl}{dt}$$
$$\frac{dl}{dt} = \frac{\cancel{2}x\frac{dx}{dt}}{\cancel{2}l}$$
$$\frac{dl}{dt} = \frac{x\, dx/dt}{l}$$

How do you find x when $l = 120$? Leave it in exact form.

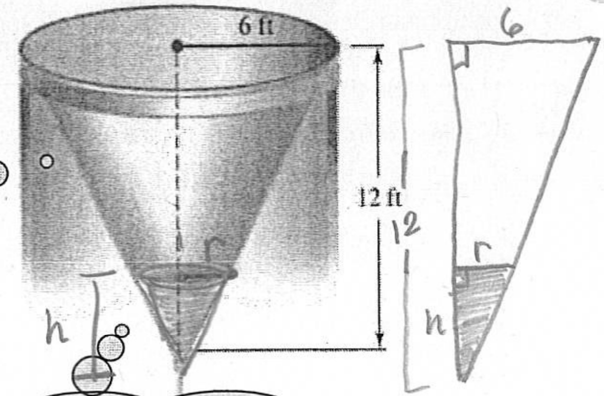
$$x^2 + 2500 = l^2$$
$$x^2 + 2500 = 120^2$$
$$x^2 + 2500 = 14400$$
$$x^2 = 11900$$
$$x = \sqrt{11900}$$
$$x = 10\sqrt{119}$$

$$\left.\frac{dl}{dt}\right|_{l=120} = \frac{10\sqrt{119}\cdot 5}{120}$$
$$= \frac{5\sqrt{119}}{12}$$
$$\approx 4.5 \text{ ft/sec}$$

$$\left.\frac{dV}{dt}\right|_{h=6} = ?$$

expl 5: Water is drained out of an inverted cone as shown. If the water level drops at the rate of 1 foot/minute, at what rate is the water (in feet^3/minute) draining from the tank when the water depth is 6 feet?

$$\frac{dh}{dt} = -1 \text{ ft/min}$$



The volume of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.

Label on this picture h to be the depth of water and r to be the radius of the top surface of the water.

We are given $\frac{dh}{dt}$. What is it? Include units. Is it positive or negative?

$$\frac{dh}{dt} = -1 \text{ ft/min}$$

Do you see similar triangles in our picture? They help us see the relationship between r and h. Use that to rewrite the volume formula in terms of h alone.

$$\frac{6}{12} = \frac{r}{h}$$

$$r = \frac{1}{2}h$$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi \left(\frac{1}{2}h\right)^2 h$$

$$V = \frac{\pi}{12} h^3$$

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{\pi}{12}h^3\right)$$

$$\frac{dV}{dt} = \frac{\pi}{12} 3h^2 \cdot dh/dt$$

$$\left.\frac{dV}{dt}\right|_{h=6} = \frac{\pi}{\cancel{12}} 3 \cdot \cancel{6^2}^{\,3} (-1)$$

$$= -9\pi$$

$$\approx -28.3 \text{ ft}^3/\text{min}$$

The volume is decreasing at the rate of 28.3 ft^3/minute.