2:00

Chapter 4 will explore using the derivative to understand a function. Where do we see a maximum or minimum value on the graph?

Calculus I
Class notes
Maxima and Minima (section 4.1)

We might have a function that describes the revenue of a company and be interested in maximizing that revenue. Say you know the trajectory of a near-Earth object and want to know its minimum distance to our planet. We start by defining absolute (a.k.a. global) maximums and minimums as seen back in algebra. The set D is the entire **domain** of the function.

> DEFINITION **Absolute Maximum and Minimum**
>
> Let f be defined on a set D containing c. If $f(c) \geq f(x)$ for every x in D, then $f(c)$ is an **absolute maximum** value of f on D. If $f(c) \leq f(x)$ for every x in D, then $f(c)$ is an **absolute minimum** value of f on D. An **absolute extreme value** is either an absolute maximum value or an absolute minimum value.

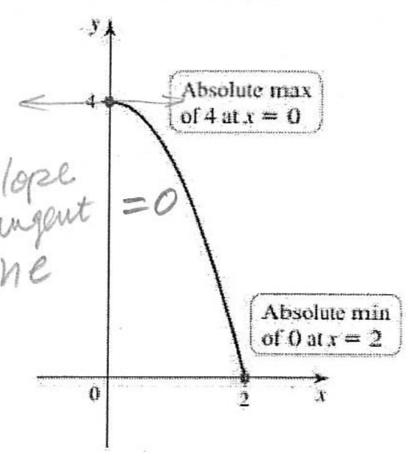
The graph shown here has both an absolute max and absolute min. Do you see why the definition above describes the points?

Singular forms: Minimum, Maximum, Extremum

Plural forms: Minima, Maxima, Extrema

Slope of tangent line $= 0$

$y = 4 - x^2$ on $[0, 2]$

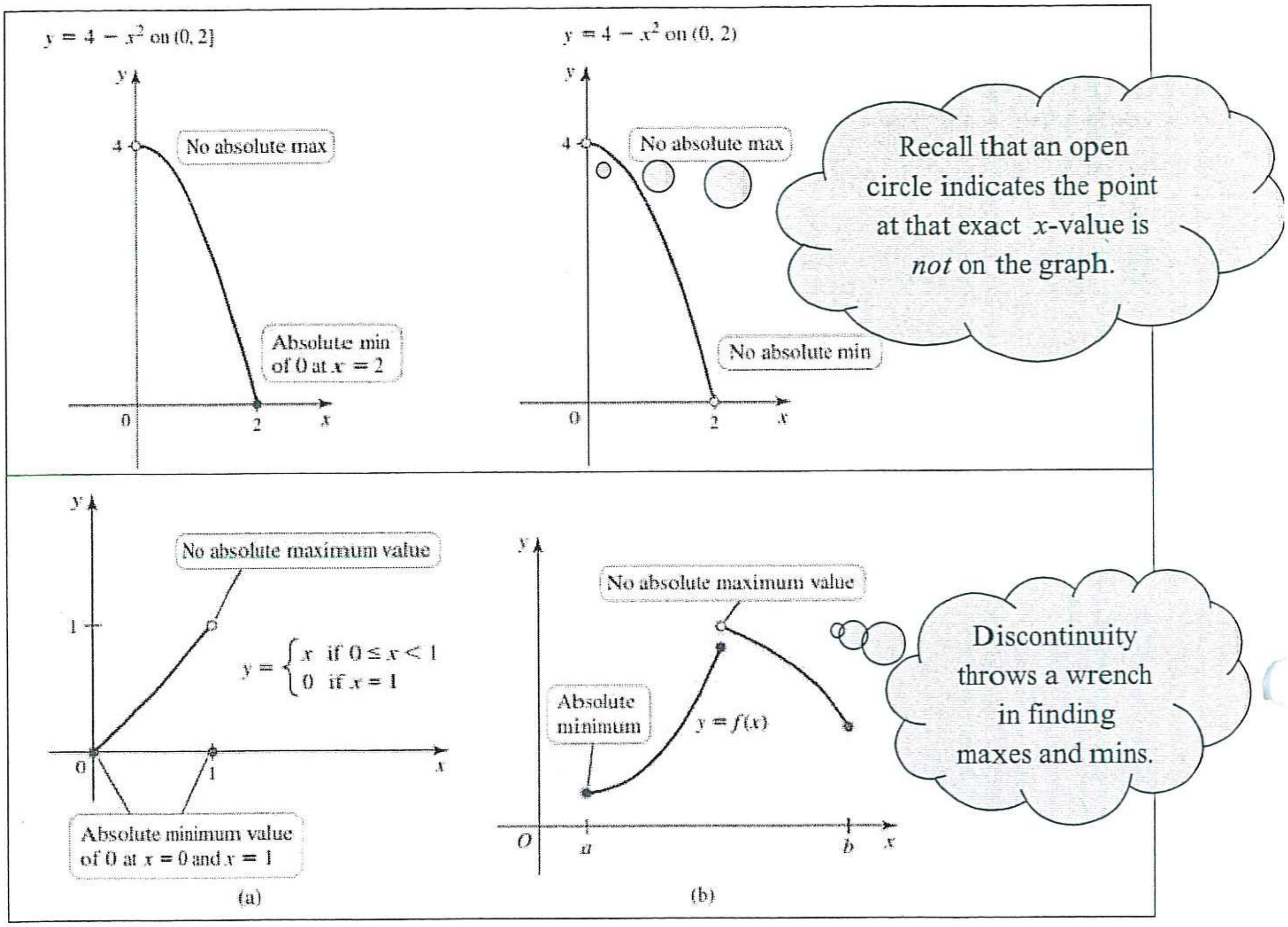


If we are asked, "Where does the extremum occur?", we give the x-value or $x = c$.

If we are asked, "What is the extremum?", we give the y-value or $f(c)$.

We have explored how these points may be where $f'(c) = 0$. You should see this for the absolute max but *not* the min above. We have much more to investigate.

Sometimes, we will encounter graphs where there is *not* an absolute max or min. Check these examples out and think about how they do *not* meet the definition given earlier.



From these examples, notice that a function being defined on a closed interval does *not* guarantee the existence of absolute maxes or mins. However, we do have this useful nugget.

> **THEOREM 4.1 Extreme Value Theorem**
> A function that is continuous on a closed interval $[a, b]$ has an absolute maximum value and an absolute minimum value on that interval.

Recall: Definition: Closed Interval: A closed interval is one where the endpoints are included in the interval such as $[0, 1]$ or $[a, b]$.

Recall: Definition: Open Interval: An open interval is one where the endpoints are *not* included in the interval such as $(0, 2)$ or (a, b).

An interval can also be half-open (half-closed).

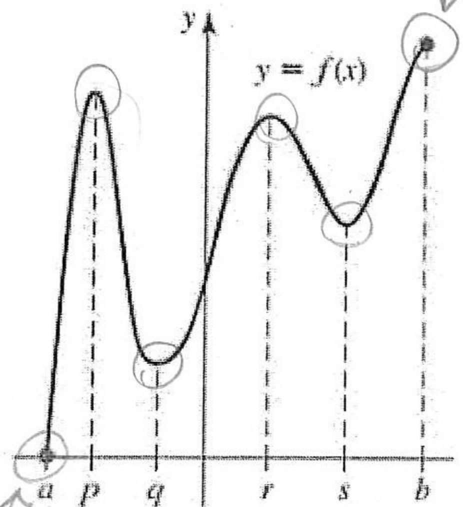
(b, f(b))

expl 1: The function pictured here is defined on the closed interval $[a, b]$.
a.) Is the function continuous on $[a, b]$? yes

b.) Does the Extreme Value Theorem apply to this function? If so, find the absolute max and min. Where do the extrema occur?

The abs min is $f(a)$. It occurs at $x=a$.

The abs max is $f(b)$. It occurs at $x=b$.



(a, f(a))

Local (Relative) Maxima and Minima:

Glance back at the picture above and consider the points where $x = p, q, r,$ and s. Notice they are the *highest or lowest points in their local area*. Hence, they fit the following definition.

> **DEFINITION Local Maximum and Minimum Values**
> Suppose c is an interior point of some interval I on which f is defined. If $f(c) \geq f(x)$ for all x in I, then $f(c)$ is a **local maximum** value of f. If $f(c) \leq f(x)$ for all x in I, then $f(c)$ is a **local minimum** value of f.

Here I can be drawn as a small circle around these interior points.

expl 2: Again, look at the picture above for the function $f(x)$. Label each y-value below as a *local* max or min or neither.

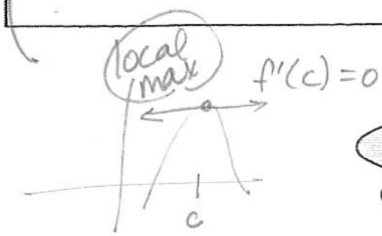
a.) $f(p)$ local max b.) $f(q)$ local min c.) $f(r)$ local max

d.) $f(s)$ local min e.) $f(a)$ neither f.) $f(b)$ neither

Careful with these last two! The value c must be an "*interior*" point. What does that mean?

Connection Back to Calculus:
We have played around with this concept a bit earlier but make it official now.

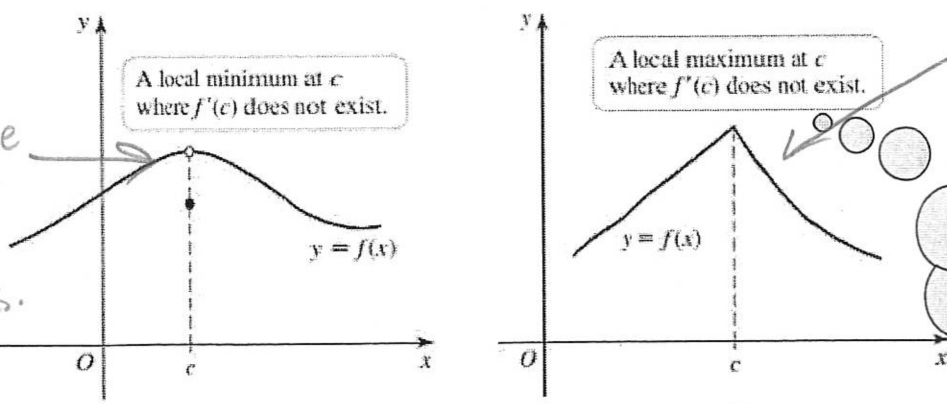
> **THEOREM 4.2 Local Extreme Value Theorem**
> If f has a local maximum or minimum value at c and $f'(c)$ exists, then $f'(c) = 0$.



Think back to the previous page's function. Would you agree that the slopes of the tangent lines at the points $p, q, r,$ and s are all zero?

Do *not* lose sight of the fact that this theorem does *not* mean that $f'(c)$ *must* exist for a local max or min to be there. Here are two counterexamples of that.

f'(c) dne because it's not continuous.



(a) (b)

f'(c) dne because of the corner

Why do we say that $f'(c)$ does *not* exist in these graphs?

So, how do we find local maxes and mins? We will use an extremely useful concept defined below.

> **DEFINITION Critical Point**
> An interior point c of the domain of f at which $f'(c) = 0$ or $f'(c)$ fails to exist is called a **critical point** of f.

In practice, critical points are *candidates* for the location of local maxes and mins. We will algebraically determine if a local extremum *actually* does exist at these critical points. We will do this later in another section.

For now, we focus on absolute maxes and mins.

PROCEDURE Locating Absolute Extreme Values on a Closed Interval

Assume the function f is continuous on the closed interval $[a, b]$.

1. Locate the critical points c in (a, b), where $f'(c) = 0$ or $f'(c)$ does not exist. These points are candidates for absolute maxima and minima.
2. Evaluate f at the critical points and at the endpoints of $[a, b]$.
3. Choose the largest and smallest values of f from Step 2 for the absolute maximum and minimum values, respectively.

In algebra class, we might have done this graphically. We now apply this procedure.

expl 3: Locate and find the values of all absolute extrema for the function $f(x) = x^2 - 10$ over the interval $[-2, 3]$.

$f'(x) = 2x$

Solve for $f'(x) = 0$ or $f'(x)$ dne.

$0 = 2x$ nowhere

$x = 0$ Critical point.

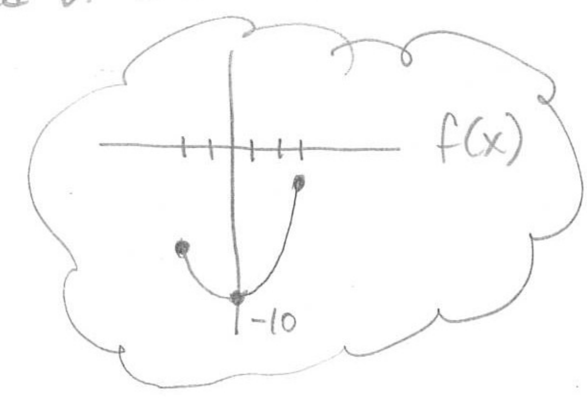
x	-2	0	3
$f(x)$	-6	-10	-1

abs min abs max.

Examples will get more complicated. Practice using an organized table for $f(x)$ values at critical points and endpoints.

The abs min is -10 and it occurs at $x=0$.

the abs max is -1 and it occurs at $x=3$.



Factor
$x^2 - 5x + 4$
$= (x-4)(x-1)$

Most problems will yield equations that are solvable through algebra; factoring will play a huge role.

expl 4: Locate and find the values of all absolute extrema for the given function over the interval $[-2, 3]$.

$f(x) = 3x^5 - 25x^3 + 60x$

$f'(x) = 15x^4 - 75x^2 + 60$

Solve $f'(x) = 0$ and $f'(x)$ dne
nowhere

$0 = 15x^4 - 75x^2 + 60$

$0 = 15(x^4 - 5x^2 + 4)$

$0 = 15(x^2 - 4)(x^2 - 1)$

$x^2 - 4 = 0$ or $x^2 - 1 = 0$

$x^2 = 4$ $\quad$ $x^2 = 1$

$x = \pm 2$ $\quad$ $x = \pm 1$

x	-2	-1	1	2	3
$f(x)$	-16	-38	38	16	234
		abs min			abs max.

You are allowed to use the calculator's **Table** function to determine $f(x)$ values. Feel free to *check* extrema by graphing.

The abs max is $f(3) = 234$ and occurs at $x = 3$.

The abs min is $f(-1) = -38$ and occurs at $x = -1$.

$\frac{d}{dr}(fg) = f'g + fg'$

$-\frac{2}{3} - 2 = -\frac{2}{3} - \frac{6}{3} = \frac{-8}{3}$ (2:00

Find $h'(r)$ and simplify. Critical points occur where $h'(r) = 0$ or is undefined.

expl 5: Locate and find the values of all absolute extrema for the given function over the interval $[-2, 2]$.

$h(r) = r^{2/3}(4 - r^2)$

$h'(r) = \frac{2}{3} r^{-1/3}(4 - r^2) + r^{2/3}(-2r)$

$= \frac{8}{3} r^{-1/3} - \frac{2}{3} r^{5/3} - 2r^{5/3}$

$h'(r) = \frac{8}{3} r^{-1/3} - \frac{8}{3} r^{5/3}$

$h'(r) = \frac{8}{3}\left(\frac{1}{r^{1/3}} - r^{5/3}\right)$

exponent rule

$a^n \cdot a^m = a^{n+m}$

$-\frac{1}{3} + 2 = \frac{-1}{3} + \frac{6}{3} = \frac{5}{3}$

$\frac{2}{3} + 1 = \frac{2}{3} + \frac{3}{3} = \frac{5}{3}$

To solve $h'(r) = 0$, many tricks will be used. ~~Remember that cubing undoes the cube root.~~ not needed

Solve $h'(r) = 0$ and $h'(r)$ undefined

$0 = \frac{8}{3}\left(\frac{1}{r^{1/3}} - r^{5/3}\right)$

$\left(0 = \frac{1}{r^{1/3}} - r^{5/3}\right) r^{1/3}$

$0 = 1 - r^2$

$r^2 = 1$

$r = \pm 1$ crit pts

$r^{1/3} = 0$

$\sqrt[3]{r} = 0$

$r = 0$ crit. point

r	-2	-1	0	1	2
$h(r)$	0	3	0	3	0
	abs min	abs max	abs MM	abs max	abs min

The abs min is 0 and occurs at $x = -2, 0, 2$.

The abs max is 3 and occurs at $x = -1, 1$.

There may be multiple mins or maxes if more than one $h(r)$ value ties for lowest or highest value.

expl 6: A sales team determines that the revenue from fruit smoothies (in dollars) is given by $R(x) = -60x^2 + 300x$ where x is the price charged for a smoothie, in dollars. The domain of this function is $0 \le x \le 5$. → $[0,5]$

a.) Find the critical points.

b.) Maximize the revenue for this smoothie.

We use calculus to answer questions we had previously done with algebra.

(a) $R'(x) = -120x + 300$

Crit pts: $R'(x) = 0$ or $R'(x)$ is undefined nowhere

$0 = -120x + 300$

$-300 = -120x$

$x = -300/-120$

$x = 2.5$ Crit pts.

(b)

x	0	2.5	5
$R(x)$	0	375	0

Abs max.

We maximize revenue if we sell smoothies for \$2.50. The maximum revenue is \$375.