Let's take what we know about functions and where they increase or are concave down or have a minimum and go wild!

Calculus I
Class notes
Putting it Together to Graph Functions (section 4.4)

In the past, these insights were absolutely necessary for graphing functions. With the advent of handheld graphers, you might think these skills are obsolete. I would argue that sometimes the calculus is easier and more precise than interpreting the complex output of a grapher. These analytic procedures also help us understand *why* the graph is the way it is.

The book gives us this set of steps so we do not leave anything behind.

Graphing Guidelines for $y = f(x)$

1. **Identify the domain or interval of interest.** On what interval(s) should the function be graphed? It may be the domain of the function or some subset of the domain.
2. **Exploit symmetry.** Take advantage of symmetry. For example, is the function *even* $(f(-x) = f(x))$, *odd* $(f(-x) = -f(x))$, or neither?
3. **Find the first and second derivatives.** They are needed to determine extreme values, concavity, inflection points, and the intervals on which f is increasing or decreasing. Computing derivatives—particularly second derivatives—may not be practical, so some functions may need to be graphed without complete derivative information.
4. **Find critical points and possible inflection points.** Determine points at which $f'(x) = 0$ or f' is undefined. Determine points at which $f''(x) = 0$ or f'' is undefined.
5. **Find intervals on which the function is increasing/decreasing and concave up/down.** The first derivative determines the intervals on which f is increasing or decreasing. The second derivative determines the intervals on which the function is concave up or concave down.
6. **Identify extreme values and inflection points.** Use either the First or Second Derivative Test to classify the critical points. Both x- and y-coordinates of maxima, minima, and inflection points are needed for graphing.
7. **Locate all asymptotes and determine end behavior.** Vertical asymptotes often occur at zeros of denominators. Horizontal asymptotes require examining limits as $x \to \pm\infty$; these limits determine end behavior. Slant asymptotes occur with rational functions in which the degree of the numerator is one more than the degree of the denominator.
8. **Find the intercepts.** The y-intercept of the graph is found by setting $x = 0$. The x-intercepts are found by solving $f(x) = 0$; they are the real zeros (or roots) of f.
9. **Choose an appropriate graphing window and plot a graph.** Use the results of the previous steps to graph the function. If you use graphing software, check for consistency with your analytical work. Is your graph *complete*—that is, does it show all the essential details of the function?

We will be using, from the previous section, Theorems 4.7 and 4.10 as well as the First and Second Derivative Tests (Theorems 4.8 and 4.11).

Here is a teeny tiny version of the summary graphic from that section.

MML questions will be asked as multiple-choice.

Recap of Derivative Properties

This section has demonstrated that the first and second derivatives of a function provide valuable information about its graph. The relationships among a function's derivatives and its extreme values and concavity are summarized in Figure 4.41.

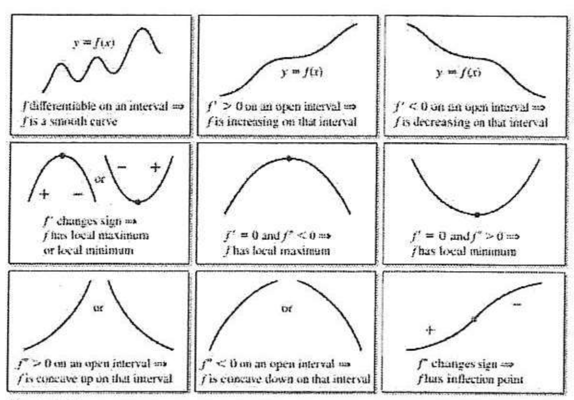


Figure 4.41

Is concave down the same as decreasing? NO

Concave down and dec — concave down but increasing (also expls: $y = \log x$ and $y = \sqrt{x}$)

expl 1: Sketch a function f with the following properties.

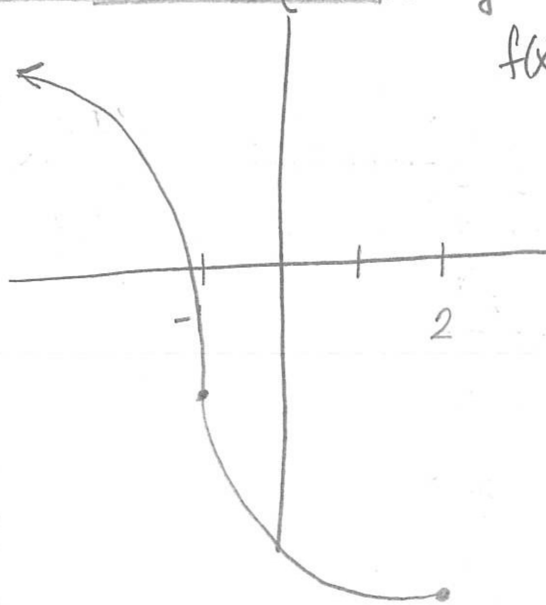
dec — concave down

$f' < 0$ and $f'' < 0$ for $x < -1$

$f' < 0$ and $f'' > 0$ for $-1 < x < 2$

Still dec. — Concave up.

Theorems 4.7 and 4.10 will help.



Symm about the origin — $y = x^3$

expl 2: Use the guidelines of the section to make a complete graph of $f(x) = x^4 - 6x^2$. We will work through the steps one-by-one.

If $f(-x) = -f(x)$, then odd.
If $f(-x) = f(x)$, then even.

Symm about y-axis — $y = x^2$

1.) What is f's domain?

$(-\infty, \infty)$

2.) Is the function even or odd?

Find $f(-x) = (-x)^4 - 6(-x)^2$
$= x^4 - 6x^2 = f(x)$

$(-x)^4 = (-x)(-x)(-x)(-x) = x^4$

So, f is even (symm about y-axis)

3.) Find $f'(x)$ and $f''(x)$.

$f'(x) = 4x^3 - 12x$

$f''(x) = 12x^2 - 12$

Find where the first and second derivatives are zero or undefined.

f' and f'' are never undefined.

4.) Find the critical points and possible inflection points.

$f'(x) = 4x^3 - 12x = 0$

$4x(x^2 - 3) = 0$

$4x = 0$ or $x^2 - 3 = 0$

$x = 0$ $\quad$ $x = \pm\sqrt{3}$

$f''(x) = 12x^2 - 12 = 0$

$12(x^2 - 1) = 0$

$x^2 - 1 = 0$

$x = \pm 1$

$\sqrt{3} \approx 1.7$

Make sign graphs for $f'(x)$ and $f''(x)$.

$f(x) = x^4 - 6x^2$

expl 2: (continued)

5.) Use test values in intervals between critical points to determine where the function is increasing or decreasing (Theorem 4.7) and concave up or down (Theorem 4.10).

$f'(x)$

Test $x=-2$ | $x=-1$ | $x=1$ | $x=2$

$f'<0$ dec | $-\sqrt{3}$ | $f'>0$ inc | 0 | $f'<0$ dec | $\sqrt{3}$ | $f'>0$ inc

$f'(x) = 4x(x^2-3)$

$f'(-2) = \text{neg} \cdot \text{pos} \rightarrow \text{neg}$

$f'(-1) = \text{neg} \cdot \text{neg} \rightarrow \text{pos}$

$f'(1) = \text{pos} \cdot \text{neg} \rightarrow \text{neg}$

$f'(2) = \text{pos} \cdot \text{pos} \rightarrow \text{pos}$

$f''(x)$

$x=-2$ | $x=0$ | $x=2$

$f''>0$ ccu | -1 | $f''<0$ ccd | 1 | $f''>0$ ccu

$f''(x) = 12(x^2-1)$

$f''(-2) = 12 \cdot \text{pos} \rightarrow \text{pos}$

$f''(0) = 12 \cdot \text{neg} \rightarrow \text{neg}$

$f''(2) = 12 \cdot \text{pos} \rightarrow \text{pos}$

ccu: concave up
ccd: concave down

6.) Use the First or Second Derivative Test to determine local extrema. Use Theorem 4.10 to find inflection points. To plot, you will need to know their y-values so calculate them now too.

First Der Test: Local max at $x=0 \rightarrow f(0) = 0^4 - 6 \cdot 0^2 = 0$

Local mins at $x = -\sqrt{3} \rightarrow f(-\sqrt{3}) = (-\sqrt{3})^4 - 6(-\sqrt{3})^2 = 9 - 6 \cdot 3 = -9$

and $x = \sqrt{3} \rightarrow f(\sqrt{3}) = -9$ (cause f is even)

Thm 4.10: f'' changes sign at $x=-1$ and 1 (inflection pts)

$x = -1 \rightarrow f(-1) = (-1)^4 - 6(-1)^2 = -5$

$x = 1 \rightarrow f(1) = 1^4 - 6(1)^2 = -5$

7.) Determine end behavior (the limits as $x \rightarrow \pm\infty$) and locate any vertical asymptotes (often where the denominator is zero).

(see section 2.5, pg 4)

$f(x) = x^4 - 6x^2$

(lead term)

end behavior:

as $x \rightarrow -\infty$, $y \rightarrow \infty$

as $x \rightarrow \infty$, $y \rightarrow \infty$.

$f(x) = x^4 - 6x^2$

expl 2: (continued)

8.) Find the x and y-intercepts.

x-int: $0 = x^4 - 6x^2$

$0 = x^2(x^2 - 6)$

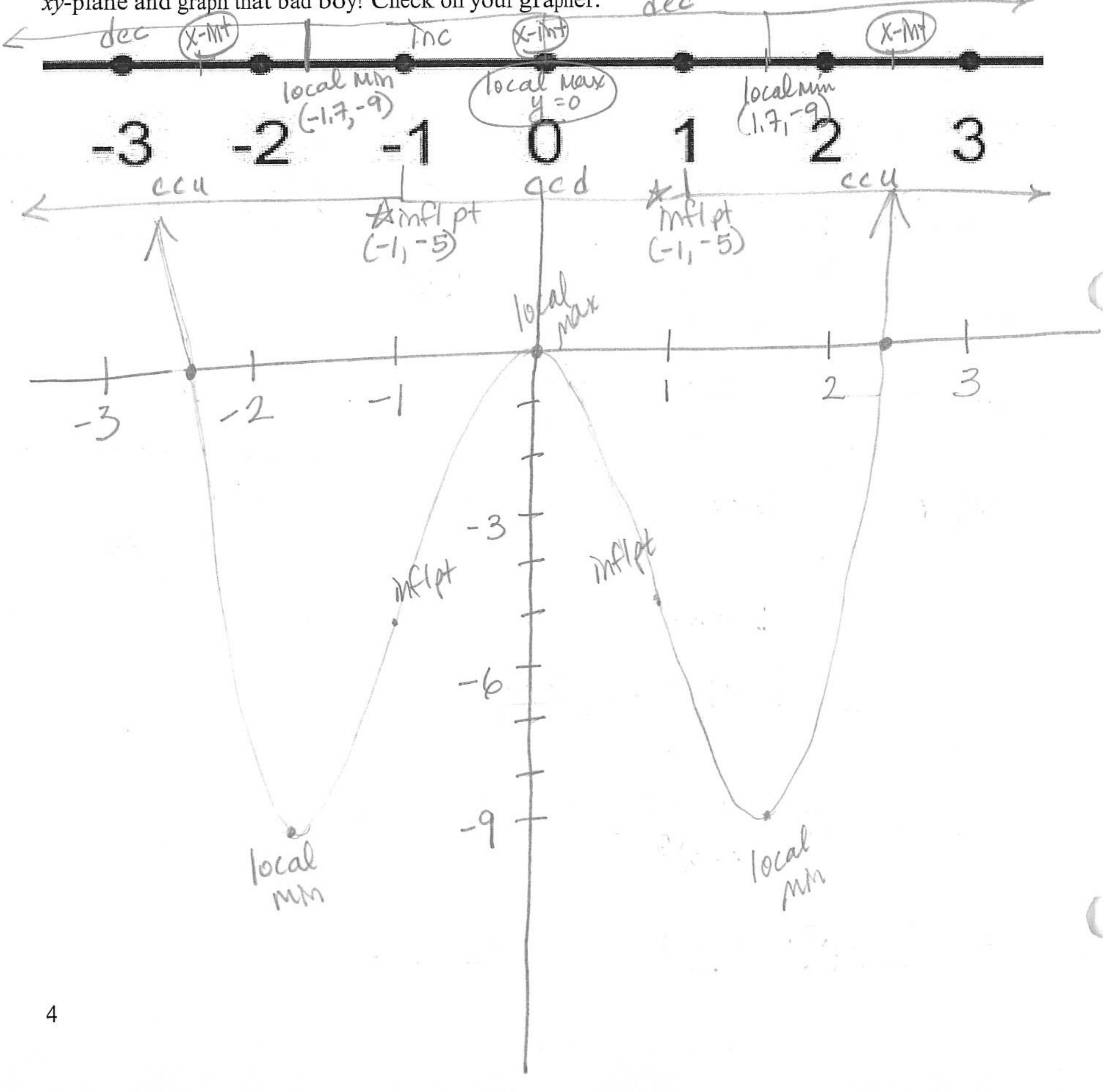
$x^2 = 0$ or $x^2 - 6 = 0$

$x = 0$ $\quad$ $x = \pm\sqrt{6}$

$x \approx \pm 2.4$

y-int: $f(0) = 0^4 - 6 \cdot 0^2 = 0$

9.) Use the real number line here to summarize all you have figured out. Plot out an appropriate xy-plane and graph that bad boy! Check on your grapher.



domain : all real numbers except what makes bottom zero.

Rational fnc

expl 3: Use the guidelines of the section to make a complete graph of $f(x)=\dfrac{x^2}{x-2}$. We will work through the steps one-by-one.

1.) What is f's domain?

all real numbers except 2.

OR $(-\infty, 2) \cup (2, \infty)$

If $f(-x)=-f(x)$, then odd.

If $f(-x)=f(x)$, then even.

2.) Is the function even or odd?

$$f(-x)=\frac{(-x)^2}{-x-2}=\frac{x^2}{-x-2}=\frac{x^2}{-1(x+2)}$$

This $f(-x)$ is neither equal to $f(x)$ nor $-f(x)$.

So, $f(x)$ is neither even nor odd.

3.) Find $f'(x)$ and $f''(x)$.

$$\frac{d}{dx}(f/g)=\frac{f'g-fg'}{g^2}$$

$$f'(x)=\frac{2x(x-2)-x^2(1)}{(x-2)^2}$$

$$=\frac{2x^2-4x-x^2}{(x-2)^2}$$

$$f'(x)=\frac{x^2-4x}{(x-2)^2}=\frac{x(x-4)}{(x-2)^2}$$

form used later

Chain Rule!

$$f''(x)=\frac{(2x-4)(x-2)^2-(x^2-4x)\cdot 2(x-2)\cdot 1}{(x-2)^4}$$

dist. prop

rearrange

$$=\frac{2(x-2)(x-2)^2-2(x-2)(x^2-4x)}{(x-2)^4}$$

$$=\frac{(x-2)[2(x-2)^2-2(x^2-4x)]}{(x-2)^4}$$

$$=\frac{(x-2)[2(x^2-4x+4)-2x^2+8x]}{(x-2)^4}$$

$$=\frac{(x-2)[\cancel{2x^2}-\cancel{8x}+8-\cancel{2x^2}+\cancel{8x}]}{(x-2)^4}$$

$$\boxed{f''(x)=\frac{8}{(x-2)^3}}$$

$f'(x) = \frac{x(x-4)}{(x-2)^2}$ $\quad$ $f''(x) = \frac{8}{(x-2)^3}$

Find where the first and second derivatives are zero or undefined.

expl 3: (continued)

4.) Find the critical points and possible inflection points.

$f' = 0 = \frac{x(x-4)}{(x-2)^2}$ $\qquad$ $f'' = 0 = 8/(x-2)^3$

(nowhere)

$0 = x(x-4)$

$x = 0$ or $x - 4 = 0$

$x = 4$

– f' is undefined at $x=2$.

– f'' is undefined at $x=2$

Make sign graphs for $f'(x)$ and $f''(x)$.

5.) Use test values in intervals between critical points to determine where the function is increasing or decreasing (Theorem 4.7) and concave up or down (Theorem 4.10).

$f'(x)$

Test	$x=-1$		$x=1$		$x=3$		$x=5$
	$f'>0$	0	$f'<0$	2	$f'<0$	4	$f'>0$
	inc		dec		dec		inc

$f'(-1) = \frac{\text{neg}\cdot\text{neg}}{\text{pos}} \to \text{pos}$

$f'(1) = \frac{\text{pos}\cdot\text{neg}}{\text{pos}} \to \text{neg}$

$f'(3) = \frac{\text{pos}\cdot\text{neg}}{\text{pos}} \to \text{neg}$

$f'(5) = \frac{\text{pos}\cdot\text{pos}}{\text{pos}} \to \text{pos}$

$f''(x)$

Test	$x=0$		$x=3$
	$f''<0$	2	$f''>0$
	ccd		ccu

$f''(0) = \frac{8}{\text{neg}} \to \text{neg}$

$f''(3) = \frac{8}{\text{pos}} \to \text{pos}$

ccd : Concave down

ccu : Concave up

$f(x) = \frac{x^2}{x-2}$

6.) Use the First *or* Second Derivative Test to determine local extrema. Use Theorem 4.10 to find inflection points. To plot, you will need to know their y-values so calculate them now too.

2nd Der Test: We'll use $c = 0, 4$ (from Step 4 above)

So, $f''(0) < 0 \to$ local max $(0,0)$ $\qquad$ $f(0) = 0^2/(0-2) = 0$

So, $f''(4) > 0 \to$ local min $(4,8)$ $\qquad$ $f(4) = 4^2/(4-2) = \frac{16}{2} = 8$

Do you remember polynomial long division?

$f(x) = \frac{x^2}{x-2}$ ← deg on top = 2

↑ deg on bottom = 1

expl 3: (continued)

7.) Determine end behavior (the limits as $x \to \pm\infty$, horizontal or oblique asymptotes) and locate any vertical asymptotes (often where the denominator is zero).

divisor

$$\begin{array}{r|l} & x+2 \leftarrow \text{quotient} \\ x-2 & x^2 \leftarrow \text{dividend} \\ & -(x^2-2x) \\ \hline & 2x \\ & -(2x-4) \\ \hline & 4 \leftarrow \text{remainder} \end{array}$$

"deg is 1"

"deg is 0"

$\boxed{\begin{array}{l}\text{Oblique asymptote: } y = x+2 \\ \text{Vertical asymptote: } x = 2\end{array}}$

If you're dividing by $x \pm a$, you can use synthetic division. It goes quicker.

8.) Find the x and y-intercepts. $f(x) = \frac{x^2}{(x-2)}$

x-int: $0 = x^2/(x-2)$

$0 = x^2$

$x = 0$

y-int: $f(0) = 0^2/(0-2) = 0$

$y = 0$

expl 3: (continued)

9.) Use the real number line here to summarize all you have figured out. Plot out an appropriate *xy*-plane and graph that bad boy! Check on your grapher.