2:00

Find the maximum height. Find the minimum cost. Find the maximum volume. Find the maximum fun!

Calculus I
Class notes
Optimization Problems (section 4.5)

We seek to find the optimum value of a function given some constraint. We might have a box with a fixed surface area and need to maximize its volume. Perhaps we are asked to find the dimensions of a structural beam that maximize strength while keeping costs as low as possible.

We will be given some function that needs optimizing (minimizing or maximizing) that is defined in two or more independent variables. This will be our **objective function**. We will also be given some condition (**constraint**) that will link the independent variables.

Recall, the extrema of a function will be found at critical points which are where its derivative is zero or does *not* exist. Hello calculus, my old friend. I've come to talk with you again.

Once again, the book gives us a useful game plan. 4.1

Guidelines for Optimization Problems

1. Read the problem carefully, identify the variables, and organize the given information with a picture.
2. Identify the objective function (the function to be optimized). Write it in terms of the variables of the problem.
3. Identify the constraint(s). Write them in terms of the variables of the problem.
4. Use the constraint(s) to eliminate all but one independent variable of the objective function.
5. With the objective function expressed in terms of a single variable, find the interval of interest for that variable.
6. Use methods of calculus to find the absolute maximum or minimum value of the objective function on the interval of interest. If necessary, check the endpoints.

We will start with an objective function (call it $f(x)$) that may have multiple independent variables. Our first job is to use the given constraint(s) to *rewrite this objective function with just one variable*. There may be harder and easier ways to do this. Then we find its derivative and go from there…

If the objective function is very nasty to differentiate, consider rewriting it in terms of another variable. Example 2 presents another method.

We'll find c such that $f'(c) = 0$ or does *not* exist. We will check $f(c)$ and $f(x)$ at endpoints to find its extrema.

4.1

$(x-h)^2+(y-k)^2=r^2$ Center (h,k) rad $=r$

$x^2+y^2=r^2$ center $(0,0)$ radius $=r$

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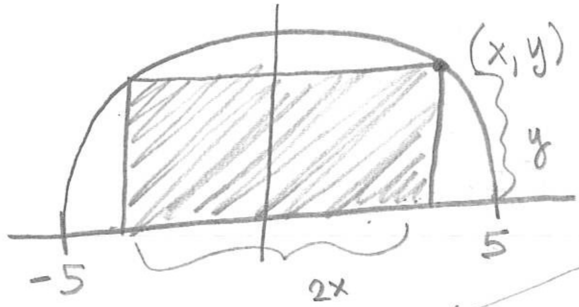
w $A=lw$

l

$x^2+y^2=25$

$y^2=25-x^2 \rightarrow y=\sqrt{25-x^2}$

expl 1: A rectangle is constructed with its base on the diameter of a semicircle with radius 5 and its two other vertices on the semicircle. What are the dimensions of the rectangle with maximum area?



Draw a picture with these shapes centered and above the x-axis.

What is the function to be maximized?

We need the area of a rectangle and the equation of a (semi)circle.

$A = 2x \cdot y$

$A = 2x\sqrt{25-x^2}$

Interval of interest is $(0,5)$

$A = \underbrace{2x}\,\underbrace{(25-x^2)^{1/2}}$

Product Rule Chain Rule/Power Rule

$A' = 2(25-x^2)^{1/2} + \cancel{2}x\left(\frac{1}{\cancel{2}}(25-x^2)^{-1/2}(-2x)\right)$

$A' = 2(25-x^2)^{1/2} - 2x^2(25-x^2)^{-1/2}$

$A' = \frac{2(25-x^2)^{1/2}}{1} - \frac{2x^2}{(25-x^2)^{1/2}}$

$= \frac{2(25-x^2)}{(25-x^2)^{1/2}} - \frac{2x^2}{(25-x^2)^{1/2}}$

$A' = \frac{50-2x^2-2x^2}{(25-x^2)^{1/2}} \rightarrow \boxed{A' = \frac{50-4x^2}{(25-x^2)^{1/2}}}$

(more room on next page)

Crit Pts occur where

$A'=0$ or undefined

when $x=5$ not feasible

Solve $A' = 0$

expl 1 work space:

$$0 = 50 - 4x^2$$
$$-50 = -4x^2$$
$$x^2 = 50/4$$
$$x = \sqrt{50/4}$$
$$x = 5\sqrt{2}/2$$

From 4.1:

x	0	$5\sqrt{2}/2$	5
$A(x)$	no rect	25 max	no rect

Find length $= 2x = 2\left(\frac{5\sqrt{2}}{2}\right) = 5\sqrt{2}$ units ≈ 7.1 units

and width $= y = \sqrt{25 - x^2}$

$$= \sqrt{25 - \left(\frac{5\sqrt{2}}{2}\right)^2} = \sqrt{25 - \frac{25 \cdot 2}{4}}$$
$$= \sqrt{25 - \frac{25}{2}}$$
$$= \sqrt{\frac{50-25}{2}} = \sqrt{\frac{25}{2}}$$

width $= \frac{5}{\sqrt{2}}$ OR $\frac{5 \cdot \sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{5\sqrt{2}}{2}$

Be sure you answer the question asked, leaving answers in exact form.

So, length of the rectangle with maximum area is $5\sqrt{2}$ units, the width is $\frac{5\sqrt{2}}{2}$ units. (The area $\frac{5\sqrt{2}}{1} \cdot \frac{5\sqrt{2}}{2} = \frac{25 \cdot 2}{2} = 25$ sq units.)

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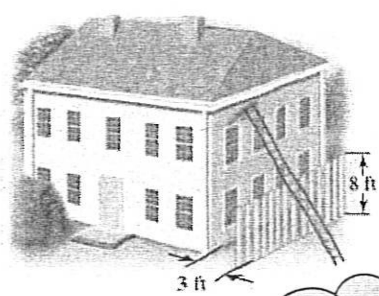
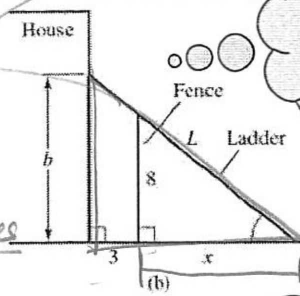
expl 2: An 8-foot tall fence runs parallel to the wall of a house at a distance of 3 feet. Find the length of the shortest ladder that extends from the ground to the house *without* touching the fence. Assume the vertical wall of the house and horizontal ground have infinite extent.



(a)

Let L, b, and x be defined as in picture. Do you see the Pythagorean theorem?

We have two similar triangles.



(b)

x = distance from fence to bottom end of ladder.

larger triangle:

$L^2 = (x+3)^2 + b^2$

Similar triangles

$\frac{b}{x+3} = \frac{8}{x}$

$bx = 8(x+3)$

$b = \frac{8(x+3)}{x}$

$b = \frac{8x+24}{x}$

$b = 8 + 24/x$

$L^2 = (x+3)^2 + (8 + 24/x)^2$

$L^2 = (x+3)^2 + (8 + 24x^{-1})^2$

Chain Rule/ Power Rule

L is a nonnegative function. Hence, L and L^2 have local extrema at the same points. We choose to minimize L^2 as it simplifies our job.

$\frac{d}{dx}(L^2) = 2(x+3)(1) + 2(8+24x^{-1})'(-24x^{-2})$

$= 2x + 6 + (-48x^{-2})(8+24x^{-1})$

Interval of interest: $0 < x < \infty$

$\frac{d}{dx}(L^2) = 2x + 6 - 384x^{-2} - 1152x^{-3}$

$\frac{d}{dx}(L^2) = \frac{2x}{1} + \frac{6}{1} - \frac{384}{x^2} - \frac{1152}{x^3}$

Ruffini Method

$= \frac{2x^4}{x^3} + \frac{6x^3}{x^3} - \frac{384x}{x^3} - \frac{1152}{x^3}$

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$\frac{d}{dx}(L^2) = \frac{2x^4 + 6x^3 - 384x - 1152}{x^3}$

factor top "factor by grouping"

$$\frac{d}{dx}(L^2) = \frac{2x^4 + 6x^3 - 384x - 1152}{x^3}$$

factoring by grouping

expl 2 work space:

$$= \frac{2x^3(x+3) - 384(x+3)}{x^3}$$

$$\frac{d}{dx}(L^2) = \frac{(x+3)(2x^3 - 384)}{x^3}$$

This is undefined at $x=0$ (not feasible)

Solve $\frac{d}{dx}(L^2) = 0 = (x+3)(2x^3 - 384)$

$x + 3 = 0$ or $2x^3 - 384 = 0$

$x = -3$ not feasible

$2x^3 = 384$

$x^3 = 192$

$x = \sqrt[3]{192}$

$= \sqrt[3]{64 \cdot 3}$

$x = 4\sqrt[3]{3}$ ft. ≈ 5.8 ft

$4^3 = 64$

$\sqrt[3]{64} = 4$

Recall, critical points occur where the derivative is 0 or does *not* exist. Describe the ladder when the derivative does *not* exist.

<u>Find L:</u>

$L^2 = (x+3)^2 + (8 + 24/x)^2$

$L^2 = (4\sqrt[3]{3} + 3)^2 + (8 + \cancel{24}^{6}/\cancel{4}\sqrt[3]{3})^2$

$L^2 \approx 224.8$

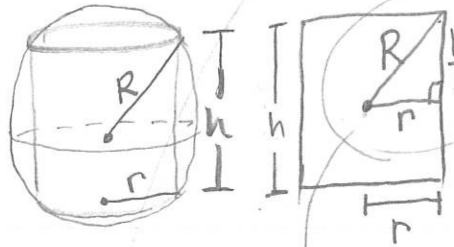
$L \approx 15$ feet

Use the First Derivative Test. Theorem 4.9 tells us that, for a continuous function, exactly one local extrema implies it's the absolute extrema.

Don't forget to find the length of the ladder. Round appropriately.

$$V = \pi r^2 h$$

expl 3: Find the height h, radius r, and volume V of a right circular cylinder with maximum volume that is inscribed in a sphere of radius R.



Draw this cylinder and sketch a sphere (with its center) around it. Note that the entire top and bottom edges of the cylinder would be touching the sphere. Label h, r, and R. Can you imagine the middle vertical plane within the cylinder?

$$R^2 = r^2 + (h/2)^2$$
$$R^2 = r^2 + h^2/4$$
$$r^2 = R^2 - h^2/4$$

We seek to maximize volume (objective function). That has two independent variables so we'll have to use the sphere constraint to rid ourselves of one.

$$V = \pi r^2 h$$
$$V = \pi (R^2 - h^2/4) h$$
$$V = \pi (R^2 h - h^3/4)$$
$$V' = \pi (R^2 - 3h^2/4)$$

Can you imagine this cylinder when $h = 0$? What about when $h = 2R$?

Interval of interest

when $h=0$, cylinder is a flat circle

no vol

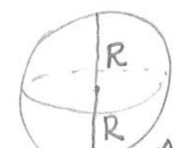
Where is V' undefined?

nowhere

Where is $V' = 0$:

$$0 = \pi (R^2 - 3h^2/4)$$
$$0 = R^2 - 3h^2/4$$
$$3h^2/4 = R^2$$

Here, $h = 2R$ cylinder is a thin segment.



no vol

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$$h^2 = \frac{4R^2}{3}$$
$$h = \sqrt{\frac{4R^2}{3}} = \frac{2R}{\sqrt{3}}$$

Our critical point will be in terms of R, the radius of the sphere.

expl 3 work space: Now, $\boxed{h = \frac{2R}{\sqrt{3}} \text{ units}}$

and $r^2 = R^2 - h^2/4$

$$= R^2 - \frac{\cancel{4}R^2}{3}\cdot\frac{1}{\cancel{4}}$$

$$= 1\cdot R^2 - \frac{1}{3}R^2$$

$$r^2 = \frac{2}{3}R^2$$

$$r = \sqrt{\frac{2}{3}R^2}$$

$$\boxed{r = \frac{\sqrt{2}}{\sqrt{3}}R \text{ units}}$$

Our answers will be in terms of *R*, the radius of the sphere.

And $V = \pi r^2 h$

$$= \pi\left(\frac{\sqrt{2}}{\sqrt{3}}R\right)^2\cdot\frac{2R}{\sqrt{3}}$$

$$= \pi\cdot\frac{2}{3}R^2\cdot\frac{2R}{\sqrt{3}}$$

$$\boxed{V = \frac{4\pi}{3\sqrt{3}}R^3 \text{ units cubed}}$$