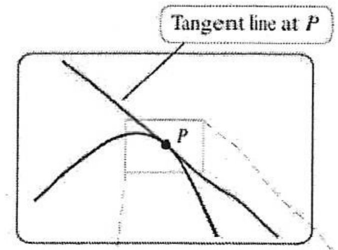
We use the tangent line to a curve to approximate a nearby function value.

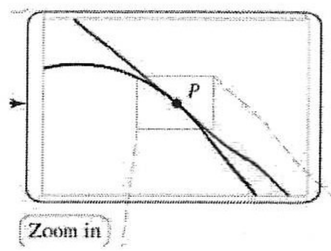
Calculus I
Class notes
Linear Approximation and Differentials (section 4.6)

Imagine the graph of a generic function and the tangent line at the point P. Do you imagine something like the picture to the right?



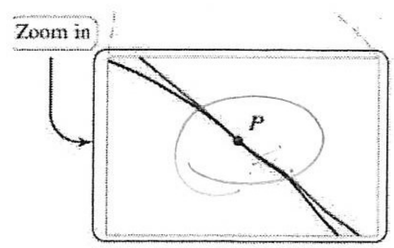
Now, in that beautiful imagination of yours, zoom in on that point. You can also zoom in your actual head. If you do it fast enough, you get a rush. Don't do it too fast or you'll hit your head.

Zoomed in, it looks like this. Zoom in even further. Go ahead, do it with your head; I know you want to.



Seriously, why not do it? No one is watching, or at least filming you, probably. Put your pencil down first.

OK, here is what it looks like when you zoom in.



Do you notice that the tangent line starts to coincide with the graph as we zoom in?

Now, here's the cool thing about this. If we need to approximate the $f(x)$ value for an x-value near this point P, we could, in fact, use the tangent line.

If our x-value was *not* near P, the tangent line would *not* give a good approximation.

Do you remember the formula for the *equation* of the tangent line of $f(x)$ at $x = a$? Write it here and solve for y.

Point Slope Formula

$$y - y_1 = m(x - x_1)$$

$$y - f(a) = m_{tan}(x - a)$$

$$y - f(a) = f'(a)(x - a)$$

$$\boxed{y = f(a) + f'(a)(x - a)}$$

DEFINITION Linear Approximation to f at a

Suppose f is differentiable on an interval I containing the point a. The **linear approximation** to f at a is the linear function

$$L(x) = f(a) + f'(a)(x - a), \quad \text{for } x \text{ in } I.$$

We introduce a new function, $L(x)$. It approximates $f(x)$ for x-values near a.

Of course, to do this, $f'(a)$ must exist.

expl 1a: Find the linear approximation to the function below at the given point a.

$f(x) = x^3 - 5x + 3$; $a = 2$ $\rightarrow$ $f(2) = 2^3 - 5\cdot 2 + 3 = 1$

$f'(x) = 3x^2 - 5$ $\rightarrow$ $f'(2) = 3\cdot 2^2 - 5 = 7$

We are asked for $L(x)$.

$$L(x) = f(2) + f'(2)(x-2)$$
$$= 1 + 7(x-2)$$
$$= 1 + 7x - 14$$
$$L(x) = 7x - 13$$

This approximates $f(x)$ values near $a = 2$.

More Exploration (Errors):

Use your function $L(x)$ to *approximate* $f(x)$ at $x = 2.1$. Then use the original function to find the *exact* value of $f(2.1)$. How did your approximation do?

$L(2.1) = 7(2.1) - 13 = 1.7$

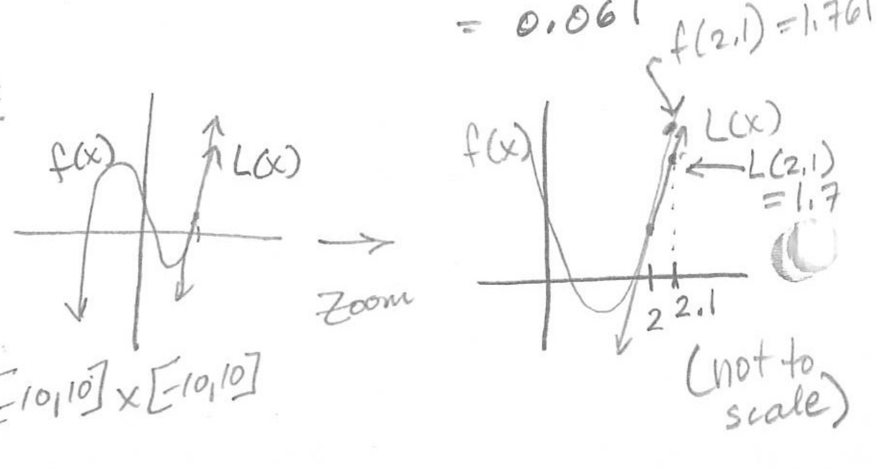
$f(2.1) = 2.1^3 - 5(2.1) + 3 = 1.761$

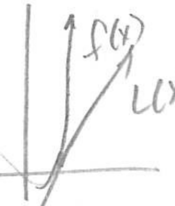
Approximation is pretty good.

We can subtract $f(2.1) - L(2.1)$ to find the **error**.

$1.761 - 1.7$
$= 0.061$

Graph the $f(x) = x^3 - 5x + 3$ and the tangent line at $a = 2$. Again, the tangent line is just the $L(x)$ we found. Zoom in and you'll see *why* we got an **underestimate**. (You should see that the tangent line is *below* $f(x)$.)





L(x) is far from f(x) when x=4.

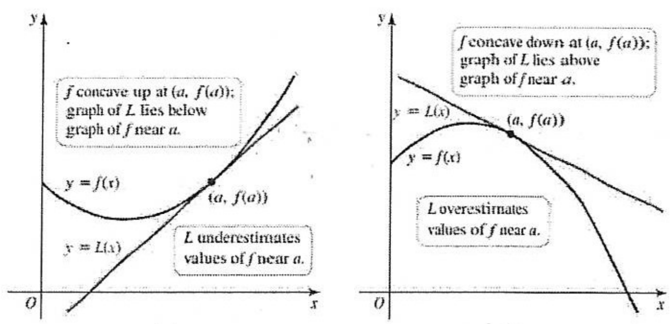
More Exploration (Values far from a):
Use your function $L(x)$ to *approximate* $f(x)$ at $x=4$. Use the original function to find the *exact* value of $f(4)$. How did your approximation do?

$L(4) = 7\cdot4-13 = 15$
$f(4) = 4^3-5\cdot4+3 = 47$

error $= 47-15 = 32$
Very bad approx.

You do not need to subtract to see that this approximation did *not* work. Return to your graph and extend $L(x)$ to $x=4$.

Concavity:
Consider these two functions. One is concave up and the other concave down at a. Notice how the tangent line for the concave up graph (left) would *always underestimate* $f(x)$. The opposite happens on the concave down graph (right). Crazy!



(a) (b)

Recall, from a previous section, concavity and $f''(x)$ are related.

It turns out that a large value of $|f''(a)|$ means that the graphs of $f(x)$ and $L(x)$ diverge quickly (resulting in large errors). However, a small value of $|f''(a)|$ means that $f(x)$ and $L(x)$ diverge less quickly (resulting in smaller errors).

When $|f''(a)|$ is large, we call that **large curvature**. When $|f''(a)|$ is small, we call that **small curvature.**

Definition: Curvature: Curvature is the degree of concavity.

expl 1b: Once again, consider $f(x) = x^3 - 5x + 3$ and its linear approximation $L(x) = 7x - 13$ when $a = 2$.

Find $|f''(2)|$. Does it seem large? Small? How can you tell?

$f'(x) = 3x^2 - 5$
$f''(x) = 6x$

$|f''(2)|$
$= |6\cdot2| = |12| = 12$

2:00

$f(x) = x^3 - 5x + 3 \rightarrow f(0) = 3$

$f'(x) = 3x^2 - 5 \rightarrow f'(0) = -5$

$f''(x) = 6x \rightarrow f''(0) = 0$

More Exploration (Second Derivatives and Errors):

expl 1c: Redo the procedure for $a = 0$. That is, find $f(0)$ and $f'(0)$. Use them to form the function $L(x)$ and *approximate* $f(x)$ at $x = 0.1$.

$L(x) = f(a) + f'(a)(x-a)$

$= f(0) + f'(0)(x-0)$

$L(x) = 3 - 5(x)$

$\boxed{L(x) = -5x + 3}$

$L(0.1) = -5(0.1) + 3$

$L(0.1) = 2.5$

$f(0.1) = 0.1^3 - 5(0.1) + 3$

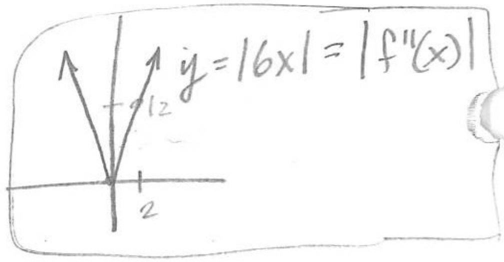
$f(0.1) = 2.501$

How did your approximation do?

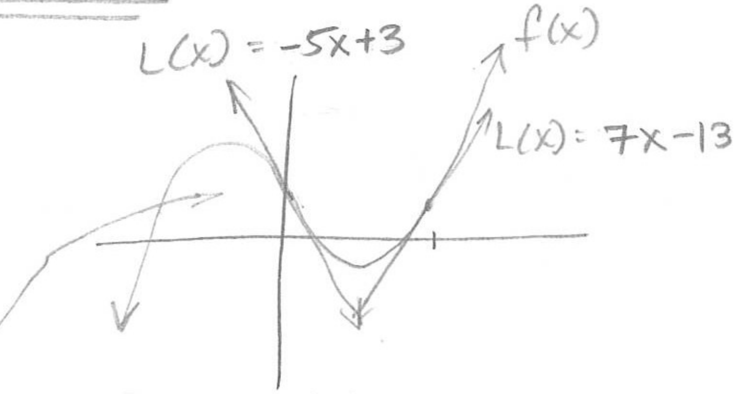
Complete the table so we can explore the connection between the error and the absolute value of the second derivative.

a	$f(a+0.1)$	$L(a+0.1)$	error	$\lvert f''(a)\rvert$
2	1.761	1.7	0.061	12
0	2.501	2.5	0.001	0

We could do this for various values of a but it would be tedious. Alternatively, let's explore the graph of $|f''(x)| = |6x|$. You could probably produce it from memory and your knowledge of transformations. Draw it now.



Do you see how the value of $|f''(x)| = |6x|$ gets larger as we get further from $x = 0$? That indicates a larger curvature on these parts of the graph of $f(x) = x^3 - 5x + 3$ which implies our approximations will be worse as we get further from zero.



Give yourself a graph of the function and both tangent lines using the window [-2.5, 2.5] x [-2.5, 10].

You will see that the tangent line at $a = 0$ coincides with $f(x)$ longer than in other places. Also of note is the fact that the tangent line is *above* $f(x)$ when $x < 0$ and *below* it when $x > 0$.

Note that the graph has a point of inflection at $x = 0$.

Estimating Change with Linear Approximations:

Let's revisit this linear approximation and use some familiar notation to denote what we've got.

$$f(x) \approx L(x) = f(a) + f'(a)(x-a)$$

$$f(x) \approx f(a) + f'(a)(x-a)$$

Subtract $f(a)$ from both sides and we see the left side as Δy. Likewise, do you see Δx? Rewrite our equation with this delta notation.

$$\underbrace{f(x)-f(a)}_{\Delta y} \approx f'(a)\underbrace{(x-a)}_{\Delta x}$$

This means that the change in y (function values) is *approximated* by the corresponding change in x-values magnified or diminished by a factor of $f'(a)$. Here, the book makes this official and gives us a picture.

Relationship Between Δx and Δy

Suppose f is differentiable on an interval I containing the point a. The change in the value of f between two points a and $a + \Delta x$ is approximately

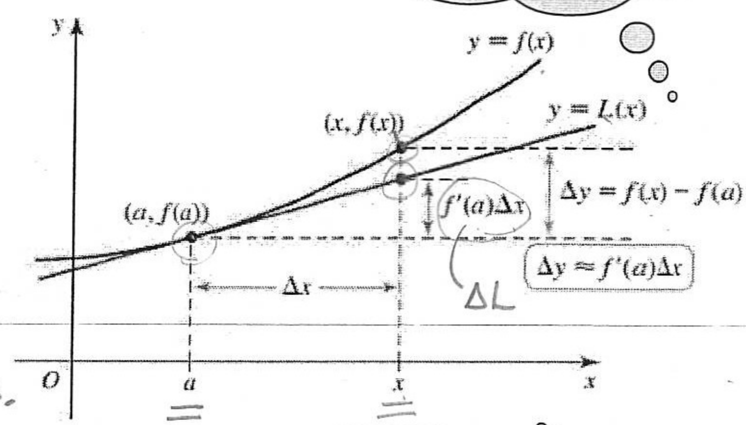
$$\Delta y \approx f'(a)\,\Delta x,$$

approx.

where $a + \Delta x$ is in I.

Look for Δx, Δy, and $f'(a)\Delta x$.

So, what good is this? It leads to an important discussion of differentials. But it will also give us insight for some interesting applications like our next example.



1 atm: equal to the avg atmos. pressure at sea level

expl 2: We are told that the atmospheric pressure at an altitude of z kilometers is given by $P(z) = 1000e^{-z/10}$ atmospheres. Approximate the change in the atmospheric pressure when the altitude increases from $z = 2$ km to $z = 2.01$ km. Do *not* forget units.

Use the formula with a change in variables.

$\Delta y \approx P'(a) \cdot \Delta z$

$a = 2$

$\Delta z = 0.01$

$\Delta y \approx P'(2)\,\Delta z$

$\Delta y \approx -100e^{-0.2}(0.01)$

$\approx -e^{-0.2} \approx -0.82$ atm

$P'(z) = 1000(-1/10)e^{-z/10}$

$P'(z) = -100e^{-z/10}$

$P'(2) = -100e^{-2/10}$

$P'(2) = -100e^{-0.2}$

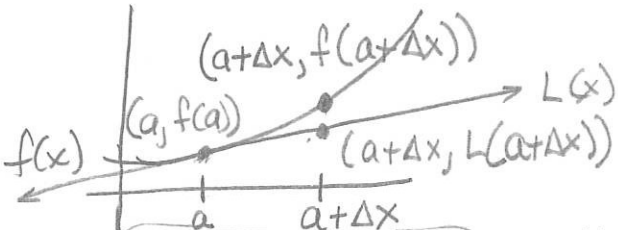
When the altitude changes from 2 km to 2.01 km, the atmos. pressure <u>decreases</u> by 0.82 atmospheres (atm).

Differentials:

We have seen these before. We simply delve deeper here.

Consider our function f that is differentiable on an interval containing $x = a$. If the x-coordinate changes from a to $a + \Delta x$, then the corresponding change in f is *exactly* $\Delta y = f(a+\Delta x) - f(a)$.

Give yourself a quick picture of a generic function with the x-values a and $a + \Delta x$ along with their function values.



On the other hand, if we use the linear approximation, $L(x) = f(a) + f'(a)(x-a)$, we would get $\Delta L = L(a+\Delta x) - L(a)$. Complete this to see what this is equal to.

$\Delta L = f(a) + f'(a)(a+\Delta x - a) - (f(a) + f'(a)(a-a))$

$\Delta L = \cancel{f(a)} + f'(a)\Delta x - f(a)$

$\Delta L = f'(a)\Delta x$

Recall, ΔL ☆ approximates Δy.

Add the tangent line at a to your picture above. Recall, this is the line $L(x)$.

Definition: Differential, *dx*: This is equal to Δx.

Definition: Differential, *dy*: This is the change in the linear approximation $dy = \Delta L = f'(a)\Delta x$. We now can see that $dy \approx \Delta y$.

Recall, this alternative notation for the derivative was given in an earlier section.

So, we can now write $dy = f'(a)\,dx$ for the point $x = a$.

For the variable point x, we would have $\boxed{dy = f'(x)\,dx}$, or rather, $\dfrac{dy}{dx} = f'(x)$.

expl 3: Express the relationship between a $\underbrace{\text{small change in } x}_{\Delta x = dx}$ and the $\underbrace{\text{corresponding change in } y}_{\Delta y \approx dy}$ in the form $\boxed{dy = f'(x)\,dx}$.

$f(x) = 1/x^3$

$f(x) = x^{-3}$

$f'(x) = -3x^{-4}$

$dy = f'(x)\,dx$

$\boxed{dy = -3x^{-4}\,dx}$

$\dfrac{dy}{dx} = -3x^{-4}$