2:00

Going back to 1669 for this oldie but goodie. I'm Casey Casem and this is a calculus method to find the root of a function.

Calculus I
Class notes
Newton's Method (section 4.8)

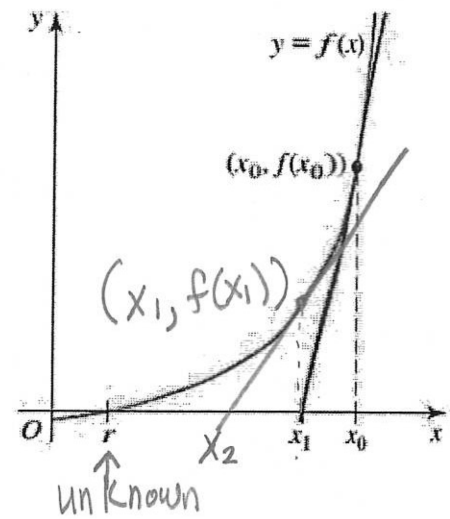
This is a procedure to find the **roots** of a function. They are also called **zeros** and are the x-values that make $f(x)$ zero. They would be the **x-intercepts** of the function's graph.

Our story starts with a function whose root (shown at $x = r$) is unknown. But we do know some calculus…

We look at some close-by x-value which is labeled as x_0.

If we draw the tangent line to the graph at this x-value, it will hit the x-axis, giving us x_1.

From this new x-value (which is x_1) we follow to the graph and find the point $(x_1, f(x_1))$. Go ahead and plot it on this picture.

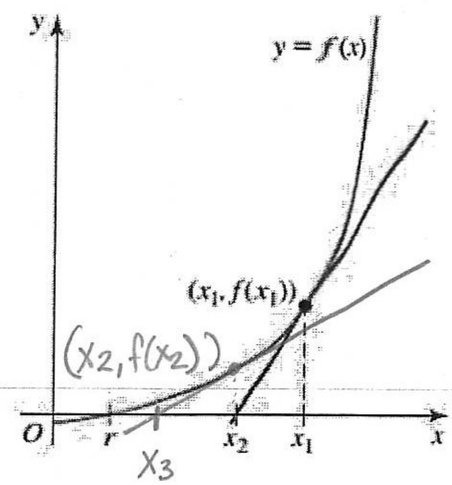


Notice that when we draw the tangent line at this new point, it hits the x-axis in yet another x-value which we call x_2.

Guess what happens next? Yes, yes, yes! Math likes to repeat itself, does it not?

So, we find the point $(x_2, f(x_2))$ and draw its tangent line. Go ahead, do that now to the right. What would you call where this line hits the x-axis?

Do you see that we are now pretty close to the root we seek? The plan is to do this until we are as close as we would want to be.



We now need an algebraic formula for these successive values of $x_1, x_2, x_3, \ldots x_n$. Think that they are each merely the x-intercept of the tangent line for the x-value that came before.

eqn of tangent line

We know $\boxed{y - f(x_n) = f'(x_n)(x - x_n)}$. Substitute the next x-intercept as $(x, y) = (x_{n+1}, 0)$ and solve for x_{n+1}.

$$0 - f(x_n) = f'(x_n)(x_{n+1} - x_n)$$

$$\frac{-f(x_n)}{f'(x_n)} = x_{n+1} - x_n$$

$$x_n - f(x_n)/f'(x_n) = x_{n+1}$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Here is our procedure as the book presents it. Did your work at the bottom of page 1 match this formula?

PROCEDURE Newton's Method for Approximating Roots of $f(x) = 0$

1. Choose an initial approximation x_0 as close to a root as possible.
2. For $n = 0, 1, 2, \ldots,$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},}$$

provided $f'(x_n) \neq 0$.

3. End the calculations when a termination condition is met.

The x_i values will start to agree to a certain number of decimal digits (digits to the right of the decimal point).

expl 1: Write the formula for Newton's Method and use the given initial approximation to compute the approximations x_1 and x_2.

$f(x) = x^2 - 6;\ x_0 = 3 \rightarrow f(x_n) = x_n^2 - 6$

$f'(x) = 2x \rightarrow f'(x_n) = 2x_n$

Find $f'(x)$. What would $f'(x_n)$ be then? Use the formula above and simplify.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = \frac{x_n}{1} - \frac{x_n^2 - 6}{2x_n} = \frac{2x_n^2}{2x_n} - \frac{x_n^2-6}{2x_n} = \frac{2x_n^2 - (x_n^2 - 6)}{2x_n}$$

$$\boxed{x_{n+1} = \frac{x_n^2 + 6}{2x_n}}$$ formula for Newton's Method.

So, $x_0 = 3$ (given):

$n=0$ $$x_{n+1} = \frac{x_n^2 + 6}{2x_n}$$

$$x_1 = \frac{x_0^2 + 6}{2 \cdot x_0} = \frac{3^2 + 6}{2 \cdot 3} = \frac{15}{6} = 2.5 \rightarrow x_1 = 2.5$$

$n=1$ $$x_2 = \frac{x_1^2 + 6}{2x_1} = \frac{2.5^2 + 6}{2 \cdot 2.5} = 2.45 \rightarrow x_2 = 2.45$$

In practice, we continue until these values start to agree.

www.stlmath.com > Courses > Math 171 > Online calculators

Newton's Method Calculator

expl 2: For the given function and initial approximation x_0, use Newton's Method to approximate a root of f. Stop calculating when two successive approximations agree to five decimal places. Round your root to five decimal places.

$f(x) = x^3 + x^2 + 1;\ x_0 = -1.5$

Use $\boxed{\wedge}$ above 6 on keyboard for exponents

We will use an online calculator like the one linked from www.stlmath.com.

$r \approx -1.46557$

(that's gonna make $f(x) = 0$)

Using Newton's Method to Find the Intersection of Two Functions:

Do you recall that you can find the intersection of two functions like $y = x^2$ and $y = 2x+3$ by setting them equal and solving?

$x^2 = 2x+3$

For these two functions, that would result in $2x+3 = x^2$ which could be solved by getting 0 on one side and factoring. Go ahead and get that 0 and you'll see what this has to do with Newton's Method (which finds the roots of functions). Don't bother solving. We have bigger fish to fry!

$x^2 - 2x - 3 = 0$

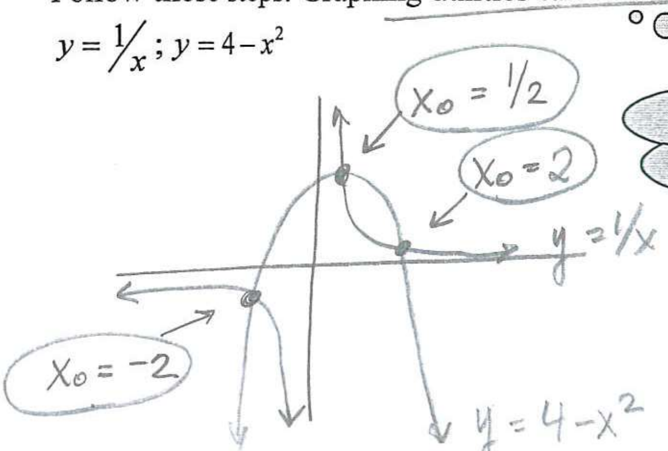
$\vdots$

Need the roots of fnc $y = x^2 - 2x - 3$ ...

Let's do one for more complicated functions that could *not* be so easily solved.

expl 3: Use Newton's Method to approximate all intersection points of this pair of curves. Follow these steps. Graphing utilities can be used to help choose initial approximations.

$y = \frac{1}{x}$; $y = 4 - x^2$



Graph both functions on the Standard Window and approximate the intersections' x-values.

Set these functions equal and rewrite to get 0 on one side. No need for fancy algebraic simplification.

$$\frac{1}{x} = 4 - x^2$$

$$x^2 + \frac{1}{x} - 4 = 0$$

Want roots of $y = x^2 + \frac{1}{x} - 4$

Use an online Newton's Method calculator, giving it the first of your guesses for x_0. Do the x_i values converge (or start to agree)? That's our root (and therefore the x-value of the intersection of the two original functions).

Do the process for each of your initial guesses.

$x_0 = -2 \rightarrow x \approx -2.11491$

$x_0 = 1/2 \rightarrow$ did not converge

$x_0 = 1/3 \rightarrow$ converging too slowly

$x_0 = 1/4 \rightarrow x \approx 0.25410$

$x_0 = 2 \rightarrow x \approx 1.86081$

If one of your initial guesses was ½, you'll see that the x_i values do *not* converge. So, look again at the graph and maybe choose 1/3. Does that work out? No, look again at the graph and try ¼…

If your initial guess is too far away from the root, this will happen.

$$\frac{d}{dx}(f/g) = \frac{f'g - fg'}{g^2}$$

Using Newton's Method to Find Maximums and Minimums:

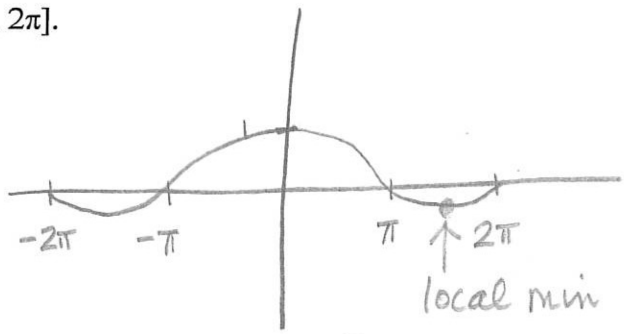
As we know, local extrema are found where $f'(x) = 0$. This means that we seek the **roots** of the *derivative* function. Let's see an example.

This is a piecewise function.

expl 4: The *sinc* function is defined as $y = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$.

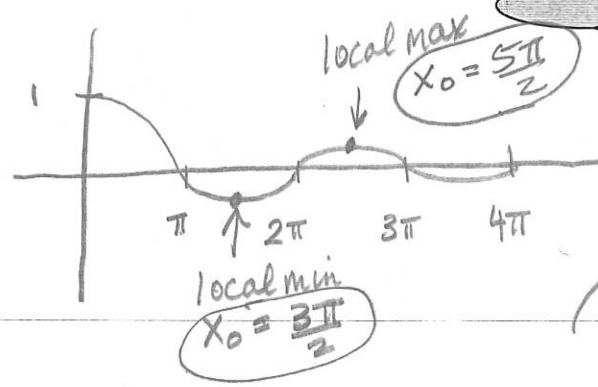
Although I would swear the book made this up, they tell us that it is used in signal-processing applications.

a.) Graph this function for $[-2\pi, 2\pi]$.



b.) Locate the *first* local minimum and the *first* local maximum of this function for $x > 0$.

Change your window to $[0, 4\pi] \times [-1, 1]$.
Set the calculator's *Xscl* to π.



Find the derivative and set it to 0. Use the online calculator. What do you choose for x_0?

$$y' = \frac{x\cos x - \sin x}{x^2}$$ Find where $y' = 0$

$x_0 = 3\pi/2 \rightarrow x \approx 4.493$

$x_0 = 5\pi/2 \rightarrow x \approx 7.725$

Do the process to find the local min and then again to find the local max.

$\hookrightarrow$ So, local min is $y \approx -0.217$ which occurs at $x \approx 4.493$.

And, local max is $y \approx 0.128$ which occurs at $x \approx 7.725$.

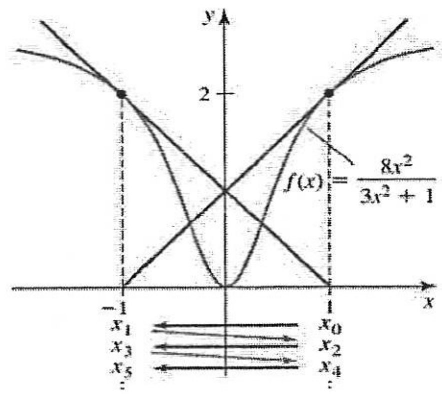
That happened in example 3.

Pitfalls of Newton's Method:

1. If the first x_0 is *not* chosen well, the values may *not* converge. Zoom in on the graph to choose a good initial value.

2. The x-value outputs may get stuck in a cycle, like alternating $-1, 1, -1, 1, -1, 1, \ldots$

Here's a picture of that, which is rare. It should work to pick another initial guess.



3. The x-value outputs may converge but *slowly*. This also happened on example 3 if you followed the thought bubble (and tried $x_0 = 1/3$).

If any of this happens to you, try another (hopefully better) initial guess. Good luck and have fun.