2:100

We know addition undoes subtraction
and square rooting undoes squaring.
What undoes taking the derivative?

Calculus I
Class notes
Antiderivatives (section 4.9)

When we differentiate a function like $g(x) = x^2$, we get $g'(x) = 2x$.

But can we go the other way? What if we were given $f(x) = 2x$ and asked to find the function whose derivative would be this $f(x)$? Would there be just one possible answer?

This is the idea of the antiderivative. Our book presents it in this traditional way.

> **DEFINITION Antiderivative**
> A function F is an **antiderivative** of f on an interval I provided $F'(x) = f(x)$, for all x in I.

Notice that we use capital F and lower case f so be careful with that. If you are *not* already in the habit of maintaining case with variables, please work on that. For the example we started with, we say that $F(x) = x^2$ is the **antiderivative** of $f(x) = 2x$ because $F'(x) = f(x)$.

So, we will be given a function which is said to be the derivative of an unknown function. It will be our job to figure out the original function. This concept will lead us to some truly lovely mathematics called **integration.** But, first things first ...

Let me give you another one. Suppose that $f(x) = 3x^2$. Can you guess its antiderivative? Or rather, *can you guess the function whose derivative is this* $f(x) = 3x^2$?

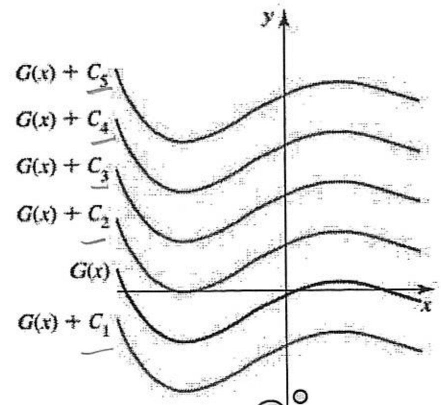
$F(x) = x^3$ $\quad\quad$ $H(x) = x^3 + \pi$
$G(x) = x^3 + 4$

If you said $F(x) = x^3$, then I say "Bravo". But is that the only possible answer? Can you think of another function whose derivative is $f(x) = 3x^2$? When you come up with others, name them H or G or the like. To be clear, any function whose derivative is $f(x) = 3x^2$ is considered an **antiderivative** of $f(x)$. We call this a family of antiderivatives.

> **THEOREM 4.14 The Family of Antiderivatives**
> Let F be any antiderivative of f on an interval I. Then *all* the antiderivatives of f on I have the form $F + C$, where C is an arbitrary constant.

$C \in \mathbb{R}$

I like the picture the book gives so here it is.



Again, consider some function $g(x)$ whose antiderivative is $G(x)$. Hence, $\frac{d}{dx}G(x) = g(x)$. (or $G'(x) = g(x)$)

Wouldn't it also be true that $\frac{d}{dx}(G(x)+c) = g(x)$ for any constant c? What derivative rule are we relying on here?

$\frac{d}{dx}(c) = 0$

When you add or subtract a real number from a function (these values of C_i), it graphs as a vertical translation.

Integration Notation:

For this new concept, we have new notation. We will write $\int f(x)dx = F(x)$. That symbol can be thought of as an elongated S for "sum"; that will be explained later. We say this is the **integral** of $f(x)$ with respect to x.

You can think of the $\int$ symbol and the dx part as two pieces of the same symbol. They say to find the antiderivative of the function $f(x)$ taking x as the **independent variable**. This function $f(x)$ is called the **integrand.**

We would write $\int 2xdx = x^2 + c,\ c \in \mathbb{R}$ since $\frac{d}{dx}\left(x^2 + c\right) = 2x$. We are undoing the act of taking a derivative, just as we learned square rooting undoes the act of squaring a number.

Do *not* leave off the dx part when writing integrals. It will be more meaningful as we add complexity.

Just as we had many shortcut rules to help us find derivatives, we have integration rules.

Look carefully and you'll see that they are simply undoing the very process we practiced with differentiation.

So, put your shoes on backwards and grab a mirror. Backwards, ho!

$\frac{d}{dx}(x^n) = nx^{n-1}$

THEOREM 4.15 Power Rule for Indefinite Integrals

$$\int x^p \, dx = \frac{x^{p+1}}{p+1} + C,$$

where $p \neq -1$ is a real number and C is an arbitrary constant.

Why would this *not* work if $p = -1$?

denom would be zero

THEOREM 4.16 Constant Multiple and Sum Rules

Constant Multiple Rule: $\int cf(x)\, dx = c \int f(x)\, dx$, for real numbers c

Difference/

Sum Rule: $\int (f(x) \pm g(x))\, dx = \int f(x)\, dx \pm \int g(x)\, dx$

For instance, $\int 2x dx = 2\int x dx$.

I would like to add a useful $\int 1dx = \int x^0 dx = x^1 + c,\ c \in \mathbb{R}$. This is direct result of the top rule but it's nice to explicitly state. I will also add $\boxed{\int \frac{1}{x} dx = \int x^{-1} dx = \ln|x| + c\ ,\ c \in \mathbb{R}}$ as it fills the gap in the first rule. We also have these gems from trig functions.

Table 4.9 Indefinite Integrals of Trigonometric Functions

1. $\frac{d}{dx}(\sin x) = \cos x \quad \Rightarrow \quad \int \cos x\, dx = \sin x + C$
2. $\frac{d}{dx}(\cos x) = -\sin x \quad \Rightarrow \quad \int \sin x\, dx = -\cos x + C$
3. $\frac{d}{dx}(\tan x) = \sec^2 x \quad \Rightarrow \quad \int \sec^2 x\, dx = \tan x + C$
4. $\frac{d}{dx}(\cot x) = -\csc^2 x \quad \Rightarrow \quad \int \csc^2 x\, dx = -\cot x + C$
5. $\frac{d}{dx}(\sec x) = \sec x \tan x \quad \Rightarrow \quad \int \sec x \tan x\, dx = \sec x + C$
6. $\frac{d}{dx}(\csc x) = -\csc x \cot x \quad \Rightarrow \quad \int \csc x \cot x\, dx = -\csc x + C$

You might also notice that these are called *indefinite* integrals. When we see the graphical meaning of a function's integral, this will make more sense. We will study both definite and indefinite integrals. For now, think of them as undoing the derivative process.

expl 1: Find all antiderivatives of the function $q(s) = \csc^2(s)$.

$\int \csc^2(s)\,ds = -\cot(s) + C, \; C \in \mathbb{R}$

Finding *all* antiderivatives is a nod to the fact that they differ by a constant. So, include $c \in \mathbb{R}$ in your answers.

$\int x^p\,dx = \dfrac{x^{p+1}}{p+1} + C$

expl 2: Determine the indefinite integral.

$\int \left(\frac{5}{t^2} + 4t^2\right) dt$ $= \int (5/t^2)\,dt + \int 4t^2\,dt$

$= \int 5t^{-2}\,dt + 4\int t^2\,dt$

$= \dfrac{5t^{-1}}{1(-1)} + \dfrac{4 \cdot t^3}{1 \cdot 3} + C$

$= -5t^{-1} + \frac{4}{3}t^3 + C, \; C \in \mathbb{R}$

$= \dfrac{-5}{t} + \dfrac{4}{3}t^3 + C, \; C \in \mathbb{R}$

As when finding derivatives, I like to rewrite terms as non-fractions and without radicals.

expl 3: Determine the indefinite integral.

$\int 6\sqrt[3]{x}\,dx$

$= \int 6x^{1/3}\,dx$

$\sqrt[3]{x} = x^{1/3}$

$= 6\int x^{1/3}\,dx$

$\frac{1}{3} + 1 = \frac{1}{3} + \frac{3}{3} = \frac{4}{3}$

$= 6 \cdot \dfrac{x^{4/3}}{4/3} + C$

$= \dfrac{18}{4} x^{4/3} + C$ OR $\boxed{\frac{9}{2}\sqrt[3]{x^4} + C}$

$= \frac{9}{2} x^{4/3} + C, \; C \in \mathbb{R}$

You can always check by finding your answer's derivative.

Check:

$\frac{d}{dx}\left(\frac{9}{2}x^{4/3} + C\right) = \frac{\cancel{9}^{3}}{\cancel{2}} \cdot \frac{\cancel{4}^{2}}{\cancel{3}} x^{1/3} + 0 = 6x^{1/3}$ ✓

expl 4: Determine the indefinite integral.

$$\int \frac{y^3 - 9y^2 + 20y}{y-4}\,dy$$

We do *not* have a rule for the integral of a quotient. However, algebra can reduce this.

$$y^3 - 9y^2 + 20y$$
$$= y(y^2 - 9y + 20)$$
$$= y(y-4)(y-5)$$

$$\int \frac{y\cancel{(y-4)}(y-5)}{\cancel{y-4}}\,dy$$

$$= \int (y^2 - 5y)\,dy$$

Constant Multiple Rule/Difference Rule

$$= \int y^2\,dy - 5\int y\,dy$$

Power Rule

$$= \frac{y^3}{3} - 5\frac{y^2}{2} + C$$

$$= \frac{1}{3}y^3 - \frac{5}{2}y^2 + C, \; C \in \mathbb{R}$$

expl 5: If $F(x) = x^2 + 3x + C$ and $F(2) = 13$, then what is C?

$$13 = 2^2 + 3\cdot 2 + C$$
$$13 = 4 + 6 + C$$
$$C = 3$$

Introduction to Differential Equations:
You will be given an equation that involves a derivative of an unknown function and asked to identify the function. This sounds like what we have been doing, doesn't it? As we have seen, that function would have to be written with some constant in place ($c \in \mathbb{R}$). Often, we will be given more information so that we can solve for that constant as well. These are called **initial value problems.**

Find $g(x)$

expl 6: Find the solution to the initial value problem.
$g'(x) = 7x^6 - 4x^3 + 12$; $g(1) = 24$

We know that $g(x) = \int g'(x)dx$. Don't forget the constant.

$g(x) = \int g'(x)\,dx$

$= \int (7x^6 - 4x^3 + 12)\,dx$

$= 7\int x^6 dx - 4\int x^3 dx + 12\int 1 dx$

Power Rule

$g(x) = \frac{\cancel{7} \cdot x^7}{\cancel{7}} - \frac{\cancel{4}x^4}{\cancel{4}} + \frac{12x^1}{1} + C$

$g(x) = x^7 - x^4 + 12x + C$

$24 = 1^7 - 1^4 + 12 \cdot 1 + C$

$24 = 1 - 1 + 12 + C$

$C = 12$

Soln:

$\boxed{g(x) = x^7 - x^4 + 12x + 12}$

expl 7: Find the function F that satisfies the following differential equation and initial conditions.
$F''(x) = 2$; $F'(0) = 5$; $F(0) = 7$

Find F' and use an initial value to find its value of c. Rinse and repeat. Can you check it?

$F'(x) = \int F''(x)\,dx$

$= \int 2\,dx$

$F'(x) = 2x + C$

$5 = 2 \cdot 0 + C$

$C = 5$

$F'(x) = 2x + 5$

$F(x) = \int F'(x)\,dx$

$= \int (2x + 5)\,dx$

$= 2\int x\,dx + \int 5\,dx$

x^0

$= \frac{2x^2}{2} + \frac{5x^{0+1}}{1} + C$

$F(x) = x^2 + 5x + C$

$7 = 0^2 + 5 \cdot 0 + C \rightarrow C = 7$

$\boxed{F(x) = x^2 + 5x + 7}$

Motion Applications:

We have seen how an object's **position** ($s(t)$), **velocity** ($v(t) = s'(t)$), and **acceleration** ($a(t) = v'(t) = s''(t)$) are related. In the past, we learned how to find the object's velocity and acceleration given its position. Here, with our new knowledge, we go the other way.

expl 8: Given the acceleration function of an object moving along a line, find the position function with the given initial velocity and position.

$a(t) = 4$; $v(0) = -3$; $s(0) = 2$

The object accelerates constantly at 4 (perhaps feet/second^2). It starts ($t = 0$) at 2 (feet) in the positive direction but it's traveling backwards at 3 (feet/second).

Keep in mind that $v(t) = \int a(t)dt$ and $s(t) = \int v(t)dt$.

$v(t) = \int a(t)dt$

$= \int 4dt$

$v(t) = 4t + c$

$-3 = 4 \cdot 0 + c$

$c = -3 \rightarrow v(t) = 4t - 3$

$s(t) = \int v(t)dt$

$= \int (4t - 3)\,dt$

$s(t) = \frac{4t^2}{2} - 3t + c$

$s(t) = 2t^2 - 3t + c$

$2 = 2 \cdot 0^2 - 3 \cdot 0 + c$

$c = 2 \longrightarrow \boxed{s(t) = 2t^2 - 3t + 2}$

Flow Rate Applications:
Whatever the function involved, if we know its rate of change (derivative), then we can use integration to find the function down to a constant. Further if we are given some point on this function (an initial point), then we can solve for that constant and know the function exactly.

expl 9: A large tank filled with water has sprung a leak (taken to be time $t = 0$). Water flows out of the tank at the rate of $Q'(t) = 0.1(100 - t^2)$ for $0 \le t \le 10$ in gallons per minute. (Presumably it took ten minutes for someone to plug it up.)

a.) Find the total amount of water that has leaked out after t minutes. This would be the function $Q(t)$. Do you see why? You may assume that $Q(0) = 0$. What does that mean?

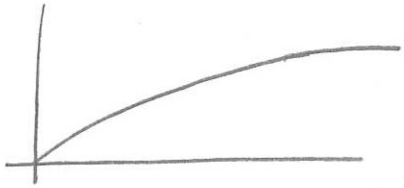
What are the units for $Q(t)$?

$$Q(t) = \int Q'(t)\,dt$$
$$= \int (10 - 0.1t^2)\,dt$$
$$Q(t) = 10t - \frac{1}{10}\frac{t^3}{3} + c$$
$$Q(t) = 10t - \frac{1}{30}t^3 + c$$
$$Q(0) = 0 = 10 \cdot 0 - \frac{1}{30}0^3 + c$$
$$c = 0 \longrightarrow \boxed{Q(t) = 10t - \frac{1}{30}t^3}$$

The total amount of water leaked by t minutes. (gallons).

b.) Graph the flow function Q for $0 \le t \le 10$ and use it to find the amount of water that leaked out in that 10 minutes.

I like to use the **Value** function under **Calculate**.



$Q(t)$

$[0, 10] \times [0, 100]$.

$Q(10) \approx 66.67$ gal has leaked out by 10 minutes.