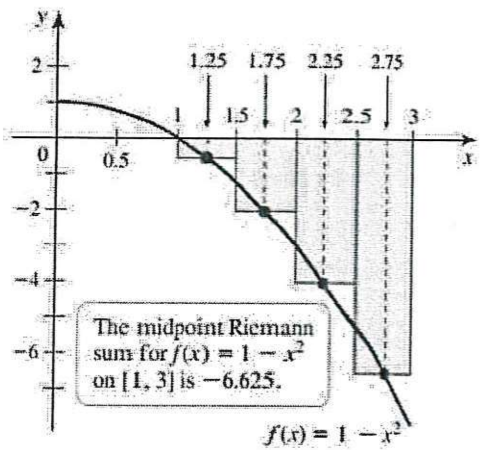
2:00

We get back to integrals and see their connection to Riemann Sums. And, what about functions that dip below the x-axis?

Calculus I
Class notes
Riemann Sums: Definite Integrals (section 5.2)

We now have the Riemann Sum that can help us estimate the area under a curve. Technically, the area we found was bound by the x-axis, wasn't it? We specifically kept to examples where the function was drawn above the x-axis. In other words, our functions were non-negative. Here, we see what it looks like when they're *not.*

Take a look at this picture of $f(x)=1-x^2$ on the interval $[1, 3]$.



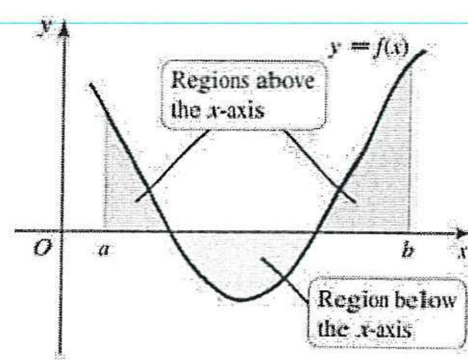
Since these function values are negative, the formula $\sum_{k=1}^{n} f\left(x_k^*\right)\Delta x$ would necessarily sum up negative numbers and end up with a negative number.

But area is *not* negative! Or, is it?

The Riemann Sum being negative does *not* mean it's useless. We just have to reimagine what it does mean. We are getting the *negative* of the area bounded by the function and the x-axis. That could be useful. Here's a definition to start making sense of this.

> **DEFINITION Net Area**
> Consider the region R bounded by the graph of a continuous function f and the x-axis between $x = a$ and $x = b$. The **net area of** R is the sum of the areas of the parts of R that lie above the x-axis *minus* the sum of the areas of the parts of R that lie below the x-axis on $[a, b]$.

Here's a picture of a function which is both below and above the x-axis.



We will count area above as *positive* and area below as *negative.*

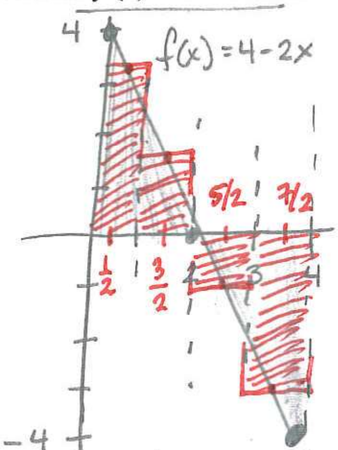
The Riemann Sum will naturally produce those signs.

$y = mx + b$

expl 1a: Consider $f(x) = 4 - 2x$. Sketch the function on the interval $[0, 4]$.

What does it look like the net area is?

0



$\Delta x = \frac{b-a}{n} = \frac{4-0}{4} = 1$

expl 1b: Approximate the net area bounded by the graph of f and the x-axis using a midpoint Riemann Sum (by hand). Use $n = 4$. Use the table below to help you organize.

x_k^*	$f(x_k^*) = 4 - 2(x_k^*)$
1/2	$4 - 2(1/2) = 3$
3/2	$4 - 2(3/2) = 1$
5/2	$4 - 2(5/2) = -1$
7/2	$4 - 2(7/2) = -3$

$\sum_{k=1}^{4} f(x_k^*) \Delta x$

$= 3 \cdot 1 + 1 \cdot 1 + -1 \cdot 1 + -3 \cdot 1$

$= 3 + 1 - 1 - 3$

$= 0$

The area should *not* surprise you.

expl 1c: Show which intervals of [0, 4] make positive and negative contributions to the net area.

Positive: subintervals [0, 1] and [1, 2]

Negative: " [2, 3] and [3, 4].

The Definite Integral:

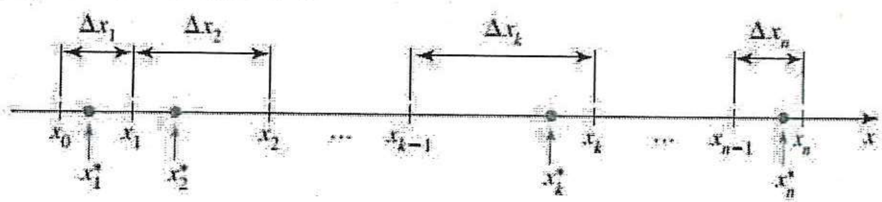
The Riemann Sum $\sum_{k=1}^{n} f\left(x_k^*\right)\Delta x$ approximates the net area. How could we find this area *exactly*?

Imagine using more and more rectangles (making n bigger and bigger). These rectangles would better fill in the area that we are after. At some point, they would be so close as to be indistinguishable from the curved area. How do we do that? Well, limits, of course!

We see that this **net area** is, in fact, *equal* to $\lim_{n\to\infty}\sum_{k=1}^{n} f\left(x_k^*\right)\Delta x$.

No longer are we saying "approximate".

We have been using Riemann Sums based on regular partitions but the truth is that the rectangles do *not* have to be of equal width. We define a **general partition** and a **General Riemann Sum**.

> **DEFINITION General Riemann Sum**
>
> Suppose $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ are subintervals of $[a, b]$ with
>
> $$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$
>
> Let Δx_k be the length of the subinterval $[x_{k-1}, x_k]$ and let x_k^* be any point in $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$.
>
> 
>
>
> If f is defined on $[a, b]$, the sum
>
> $$\sum_{k=1}^{n} f(x_k^*)\Delta x_k = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n$$
>
> is called a **general Riemann sum for f on $[a, b]$.**

This describes a **general partition.**

This is the same except that it allows for each width to be different.

If you want a left, right, or midpoint Riemann Sum, then choose x_k^* accordingly.

Now, as n gets bigger and bigger, the widths of these rectangles would get smaller and smaller, wouldn't they? We are going to say that these widths approach 0 as n approaches infinity.

If we compare all of the rectangles' widths, there would be one that is the biggest. Let's define Δ as that maximum value for widths. And this Δ would approach 0 as n approaches infinity.

We are ready…

The Definite Integral:

DEFINITION Definite Integral

A function f defined on $[a, b]$ is **integrable** on $[a, b]$ if $\lim\limits_{\Delta \to 0} \sum\limits_{k=1}^{n} f(x_k^*)\Delta x_k$ exists and is unique over all partitions of $[a, b]$ and all choices of x_k^* on a partition. This limit is the **definite integral of f from a to b**, which we write

$$\int_a^b f(x)\,dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

So, the definite integral of a function from a to b is the *exact* area bounded by the function and the x-axis. However, we consider area below the x-axis to be *negative.*

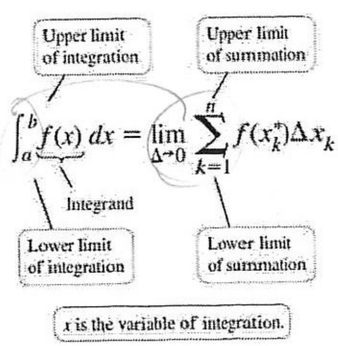
Indefinite Versus Definite Integrals:

It is important to note that the indefinite integrals we saw earlier are very much related but are *not* considered to be areas. An indefinite integral is an antiderivative of a function. In other words, the integral of a function is another function whose derivative gives us the original function.

On the other hand, a definite integral gives us this net area. We will see, in the next section, how the indefinite and definite integrals are related. And, wow, just wow. Seriously, just you wait...

Definite Integral Notation:

It does help to see why the notation is the way it is. The left and right sides of this equation are analogous.



The sum on the right, as $\Delta \to 0$, becomes a sum with an infinite number of terms, denoted by the integral sign $\int$ on the left. This was designed to look like an elongated S for sum. The **limits of integration,** a and b, also match with $k = 1$ and n from the right.

Recall that the $f(x)$ part of the integral is called the **integrand** and that you should always think of the $\int$ and the dx as two pieces of the same symbol. This dx tells us that the independent variable is x. (A moldering pile of bones, Gottfried Wilhelm Leibniz, invented this notation. Albeit, he was moldering less in 1675 when he introduced the notation.)

Evaluating Definite Integrals:

You might have noticed the definition of definite integral used the term integrable. Most of the functions we encounter will be integrable.

> **THEOREM 5.2 Integrable Functions**
> If f is continuous on $[a, b]$ or bounded on $[a, b]$ with a finite number of discontinuities, then f is integrable on $[a, b]$.

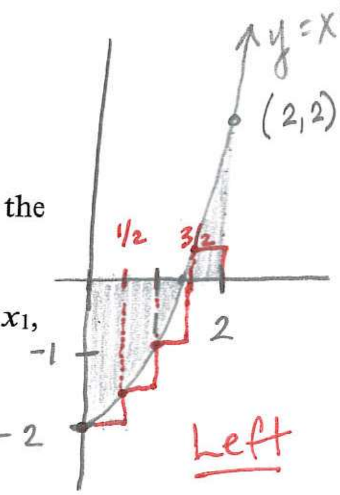
expl 2: For the integral below, do the following.

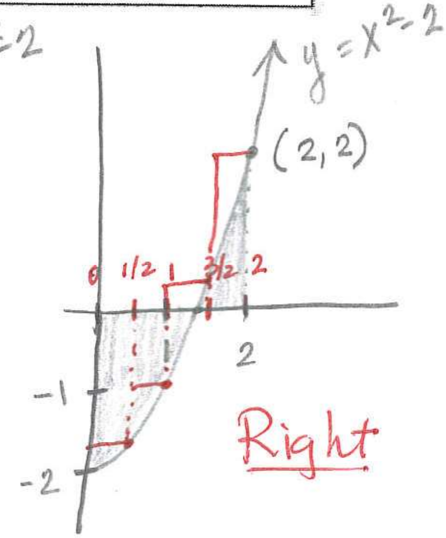
$$\int_0^2 \left(x^2 - 2\right) dx$$

a.) Sketch two graphs of the integrand on the interval of integration. Shade the areas bounded by the x-axis.

$[0,2]$

b.) Calculate Δx and the grid points x_0, x_1, x_2, … x_n. Use $n = 4$. Label them on your graphs. $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$





$X_0 = 0 \quad X_1 = 1/2 \quad X_2 = 1 \quad X_3 = 3/2 \quad X_4 = 2$

c.) Find and illustrate on your graphs both the left and right Riemann Sums. Also, tell which underestimates and which overestimates the area under the curve. Use the table below to help you organize.

x_k^*	$f(x_k^*) = (x_k^*)^2 - 2$
0	$f(0) = 0^2 - 2 = -2$
$1/2$	$f(\frac{1}{2}) = (\frac{1}{2})^2 - 2 = \frac{1}{4} - 2 = -1.75$
1	$f(1) = 1^2 - 2 = -1$
$3/2$	$f(3/2) = (\frac{3}{2})^2 - 2 = \frac{9}{4} - 2 = 0.25$
2	$f(2) = 2^2 - 2 = 2$

Left RS: $\sum_{k=1}^{4} f(x_k^*) \Delta x$

$= -2(\frac{1}{2}) + -1.75(\frac{1}{2}) + -1(\frac{1}{2}) + 0.25(\frac{1}{2})$

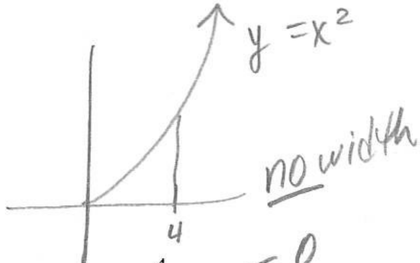
Left RS $= -2.25$ underestimated area bounded by curve

The left RS uses the first four grid points. The right RS uses the last four.

Right RS: $\sum_{k=1}^{4} f(x_k^*) \Delta x$

$= -1.75 \cdot (\frac{1}{2}) + -1(\frac{1}{2}) + 0.25(\frac{1}{2}) + 2(\frac{1}{2}) = -0.25$ Right RS (overestimated area)

(5.2)



Properties of Definite Integrals:

Some of these are common sense. Like, what do you think $\int_4^4 x^2 dx$ is? $= 0$ Think about the area that this would find. You can draw some pictures to understand most of these.

Table 5.4 Properties of definite integrals

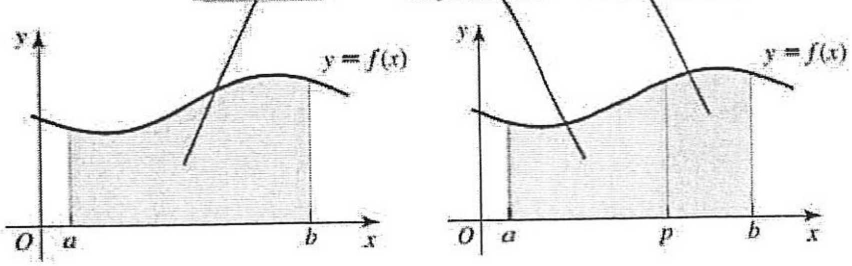
Let f and g be integrable functions on an interval that contains a, b, and p.

1. $\int_a^a f(x)\,dx = 0$ Definition
2. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ Definition
3. $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
4. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, for any constant c
5. $\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx$
6. The function $|f|$ is integrable on $[a, b]$, and $\int_a^b |f(x)|\,dx$ is the sum of the areas of the regions bounded by the graph of f and the x-axis on $[a, b]$.

$\int_p^b f(x)dx = \int_a^b f(x)dx - \int_a^p f(x)dx$

Here's a picture for one of these gems. Guess which one! Fill in the labels I have blanked out.

$\boxed{\int_a^b f(x)dx} = \boxed{\int_a^p f(x)dx} + \boxed{\int_p^b f(x)dx}$



There are more lovely pictures in the book.

I like to write smaller limit on bottom.

expl 3: You are given that $\int_1^4 f(x)dx = 8$ and $\int_1^6 f(x)dx = 5$. Find the following.

a.) $\int_1^4 3f(x)dx$

Prop4

$= 3\int_1^4 f(x)dx$

$= 3 \cdot 8$

$= 24$

b.) $\int_4^1 f(x)dx$

Prop2

$= -\int_1^4 f(x)dx$

$= -8$

c.) $\int_6^4 12f(x)dx$

Prop4

$= 12\int_6^4 f(x)dx$

Prop 2

$= -12\int_4^6 f(x)dx$

prop5

$= -12\left(\int_1^6 f(x)dx - \int_1^4 f(x)dx\right)$

$= -12(5-8)$

$= -12(-3)$

$= 36$

expl 4: You are given that $\int_0^1 (x^3 - 2x)\,dx = -3/4$. Find $\int_0^1 (10x - 5x^3)\,dx$.

How are $x^3 - 2x$ and $10x - 5x^3$ related?

$-5(x^3 - 2x) \stackrel{?}{=} 10x - 5x^3$ ✓

$$\int_0^1 (10x - 5x^3)\,dx$$

$$= \int_0^1 -5(x^3 - 2x)\,dx$$

$$= -5 \underbrace{\int_0^1 (x^3 - 2x)\,dx}_{\text{given}} = \frac{-5}{1}\left(-3/4\right) = \frac{15}{4} = 3.75$$

decimal

$3\,3/4$

expl 5: For the Riemann Sum given here, identify f and express the limit as a definite integral.

$\lim\limits_{\Delta \to 0} \sum\limits_{k=1}^{n} \left(7 + x_k^{*2}\right) \Delta x_k$ on the interval [–2, 2]

$$\int_a^b f(x)\,dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f\left(x_k^*\right) \Delta x_k$$

fnc f: $f(x) = 7 + x^2$

integral: $\int_{-2}^{2} (7 + x^2)\,dx$

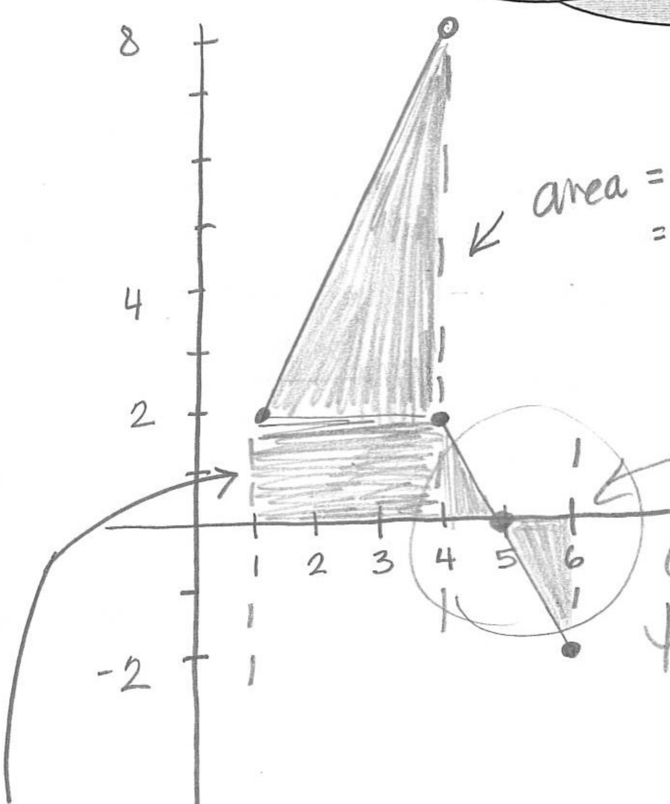
Integrals of Piecewise Functions:
An exercise in the book gives us the following.

Suppose f is continuous on the intervals $[a, p]$ and $[p, b]$ where $a < p < b$ with a finite jump at p. It is true that $\int_a^b f(x)dx = \int_a^p f(x)dx + \int_p^b f(x)dx$.

expl 6: Use geometry to find $\int_1^6 f(x)dx$ for the function $f(x) = \begin{cases} 2x, & 1 \le x < 4 \\ 10-2x, & 4 \le x \le 6 \end{cases}$.

Draw a nice big picture. Remember we seek the area bounded by f and the x-axis. You will see four geometric shapes form this area.

Take areas below the x-axis to be negative.



area $= \frac{1}{2}bh$
$= \frac{1}{2}(3)(6)$
$= 9$

$f(1) = 2 \cdot 1 = 2$
$f(4) = 2 \cdot 4 = 8$
plot it as open circle

cancel out cause they're the same size (but really other rule takes over) but one is above x-axis and one is below

$f(4) = 10 - 2(4)$
$= 2$

$f(6) = 10 - 2(6)$
$= -2$

area $= lw$
$= 2 \cdot 3 = 6$

So, net area is $\int_1^6 f(x)dx = 15$.