2:00

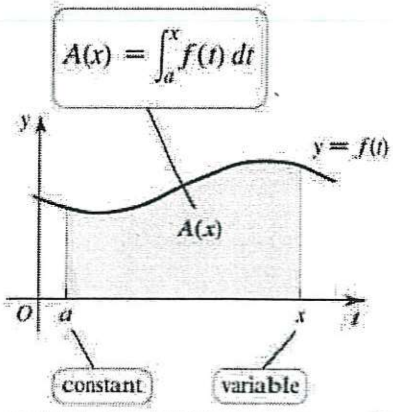
Some theorems get all the luck in naming. This one really does live up to its name. We connect antiderivatives and definite integrals.

Calculus I
Class notes
Fundamental Theorem of Calculus (section 5.3)

We know that a function's definite integral is the area under the curve, bounded by the x-axis.

We introduce the **area function** which just gives a name to that concept.

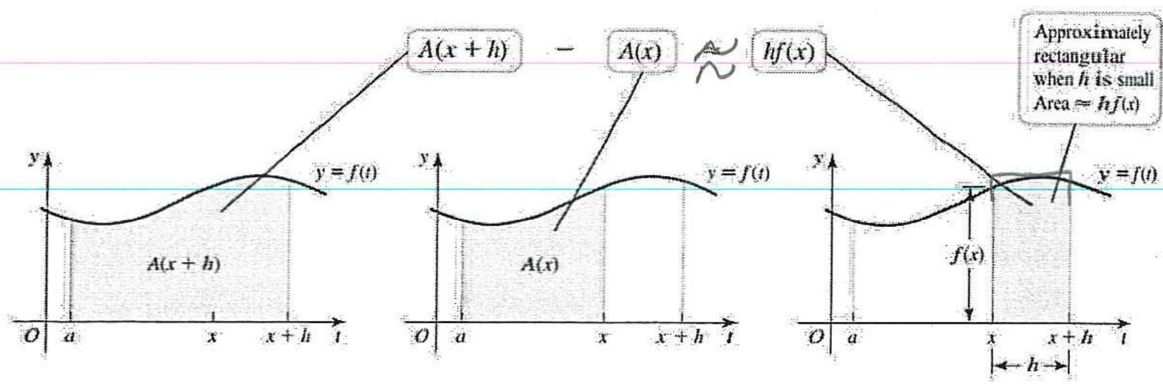
It is defined below and here is a picture.



> DEFINITION **Area Function**
>
> Let f be a continuous function, for $t \geq a$. The **area function for f with left endpoint a** is
>
> $$A(x) = \int_a^x f(t)\,dt,$$
>
> where $x \geq a$. The area function gives the net area of the region bounded by the graph of f and the t-axis on the interval $[a, x]$.

We will start with a justification of our FUN-damental theorem.

Consider these pictures which use the basic idea that you can subtract one area from another, leaving the leftover part.



We are making the assumption (*not* necessarily evident from the right-most picture) that h is small. In fact, what happens when we take h to approach 0? Does that sound familiar?

2:00 What if $f(x) = x^2$?
$\lim_{h \to 0} x^2 = x^2$

We will assume $h > 0$ but the argument is similar where $h < 0$.

We start with the equation $A(x+h) - A(x) \approx h \cdot f(x)$. Divide both sides by h and observe that as h tends toward 0 (how do you write that?), the approximation improves. In fact, we can replace the approximately equal sign with an equal sign.

$$\frac{A(x+h) - A(x)}{h} \approx \frac{h f(x)}{h}$$

What is $\lim_{h \to 0} f(x)$?

Does the left side look familiar?

$$\underbrace{\lim_{h \to 0} \frac{A(x+h) - A(x)}{h}} \approx \lim_{h \to 0} f(x)$$

Remember $A(x) = \int_a^x f(t)dt$.

$$A'(x) \approx f(x)$$

So, according to 4.9, A is an antiderivative of f.

This argument gives us the first part of the Fundamental Theorem of Calculus.

THEOREM 5.3 (PART 1) Fundamental Theorem of Calculus
If f is continuous on $[a, b]$, then the area function

$$A(x) = \int_a^x f(t)\, dt, \quad \text{for} \quad a \le x \le b,$$

is continuous on $[a, b]$ and differentiable on (a, b). The area function satisfies $A'(x) = f(x)$. Equivalently,

$$A'(x) = \frac{d}{dx} \int_a^x f(t)\, dt = f(x),$$

which means that the area function of f is an antiderivative of f on $[a, b]$.

So, this area function is an antiderivative of the function $f(x)$. That is definitely good to know but we are after the main event. What does this tell us about finding definite integrals?

from 4.9

Let's say that we have another antiderivative of f. We'll call this one $F(x)$. Do you remember that any two antiderivatives differ by a constant? How would you write that A and F are two antiderivatives that differ by a constant? Isolate $F(x)$.

$A(x) - F(x) = C$ for some $C \in \mathbb{R}$

We will consider x-values such that $a \le x \le b$.

$F(x) = A(x) - C$

Find $F(b) - F(a)$.

So, $F(b) = A(b) - C$
and $F(a) = A(a) - C$

Do you remember what $A(a)$ would have to be?

$A(a) = \int_a^a f(t)\,dt = 0$

So, $F(b) - F(a) = A(b) - C - (A(a) - C)$

$= A(b) - C - A(a) + C$

$= A(b) - A(a)$

Lastly, $A(b) = \int_a^b f(x)dx$.

So, $F(b) - F(a) = \int_a^b f(x)\,dx$

Do you see that this gives us a way to calculate $\int_a^b f(x)dx$? That is the second part of our theorem.

THEOREM 5.3 (PART 2) Fundamental Theorem of Calculus
If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Shorthand notation for $F(b) - F(a)$ is $F(x)\Big|_a^b$.

So, in short, to find the value of a definite integral, we find the integral (as we saw it before) as an antiderivative. We then evaluate the antiderivative at the limits of integration and subtract (top minus bottom).

Voilà! We have the area under the function's curve, bounded by the x-axis.

We have come full circle back to the antiderivatives where we began.

Let's practice. I will refer to the theorem as FTC.

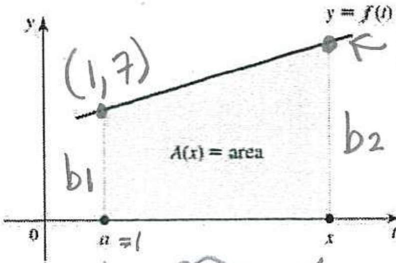
Method #1: FTC: $A(x) = \int_1^x (2t+5)\,dt = (t^2+5t)\Big|_1^x$

$= x^2+5x-(1^2+5\cdot 1) = x^2+5x-6$

So, $A(x) = x^2+5x-6$.

Handout: Paul Dawkins Calculus Cheat Sheet:
Once again, Paul gives us a concise summary of all we know so far and then some.

expl 1: For the function $f(t) = 2t+5$ shown to the right, find the area function $\boxed{A(x) = \int_a^x f(t)\,dt}$. Verify that $A'(x) = f(x)$. Assume $a = 1$.



$\Rightarrow b_1 = 7$
$b_2 = 2x+5$

Method #2: Area of trapezoid:

$A(x) = \frac{1}{2}h(b_1+b_2)$

$= \frac{1}{2}(x-1)(7+2x+5)$

$= \frac{1}{2}(x-1)(2x+12)$

$= \frac{1}{2}(2x^2-2x+12x-12)$

$= \frac{1}{2}(2x^2+10x-12)$

$\boxed{A(x) = x^2+5x-6}$ — Same as above! Whoohoo!!

The area of a trapezoid is $A = \frac{1}{2}h(b_1+b_2)$ where h is the distance between parallel bases b_1 and b_2.

expl 2: Use FTC to find the following.

a.) $\int_1^9 \frac{2}{\sqrt{x}}\,dx = \int_1^9 2x^{-1/2}\,dx = 2\int_1^9 x^{-1/2}\,dx$

Think antiderivatives now.

$= 2\cdot 2x^{1/2}\Big|_1^9$

$= 4x^{1/2}\Big|_1^9 = 4\sqrt{x}\Big|_1^9$

$= 4\sqrt{9} - 4\sqrt{1} = 12-4 = 8$

b.) $\int_0^4 3t(t+2)\,dt = \int_0^4 (3t^2+6t)\,dt$

$= (t^3+3t^2)\Big|_0^4$

$= 4^3+3\cdot 4^2 - 0$

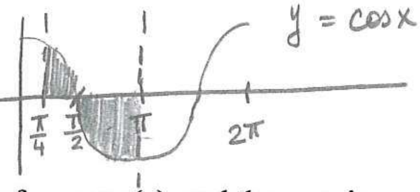
$= 64+48$

$= 112$

$\int 6t\,dt$
$= 6\int t\,dt$
$= \frac{6t^2}{2} = 3t^2$

$\int 3t^2\,dt = 3\int t^2\,dt$ Power Rule
$= \cancel{3}\cdot\frac{t^3}{\cancel{3}}$
$= t^3$

$\cancel{\int_{\pi/4}^{\pi} \cos x\,dx}$ Can't just do this cause some area is below x-axis (negative)



expl 3: Find the area of the region bounded by the graph of $y = \cos(x)$ and the x-axis on the interval $\left[\pi/4, \pi\right]$. Start with a graph.

$$\int_{\pi/4}^{\pi/2} \cos x\,dx + \left| \int_{\pi/2}^{\pi} \cos x\,dx \right|$$

They want the area all added up. But the integral won't give us that directly, will it? We will have to be creative.

OR $\int_{\pi/4}^{\pi/2} \cos x\,dx + \int_{\pi}^{\pi/2} \cos x\,dx$ OR $\int_{\pi/4}^{\pi/2} \cos x\,dx - \int_{\pi/2}^{\pi} \cos x\,dx$

$$\int_{\pi/4}^{\pi/2} \cos x\,dx + \left| \int_{\pi/2}^{\pi} \cos x\,dx \right|$$

$$= \sin x \Big|_{\pi/4}^{\pi/2} + \left| \sin x \Big|_{\pi/2}^{\pi} \right|$$

$$= \sin\left(\tfrac{\pi}{2}\right) - \sin\left(\pi/4\right) + \left| \sin(\pi) - \sin\left(\pi/2\right) \right|$$

$$= 1 - \frac{1}{\sqrt{2}} + |0-1| = 1 - \frac{1}{\sqrt{2}} + 1 = \boxed{2 - \frac{1}{\sqrt{2}}}$$ OR

You may be asked to rationalize denoms

$$\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

So, answer would be $2 - \frac{\sqrt{2}}{2}$.

expl 4: Simplify the expression.

$$\frac{d}{dx}\int_x^4 \left(t^3 + 5\right)dt$$

$f(t) = t^3 + 5$

Remember $A'(x) = \dfrac{d}{dx}\displaystyle\int_a^x f(t)dt = f(x)$.

But what is different here?

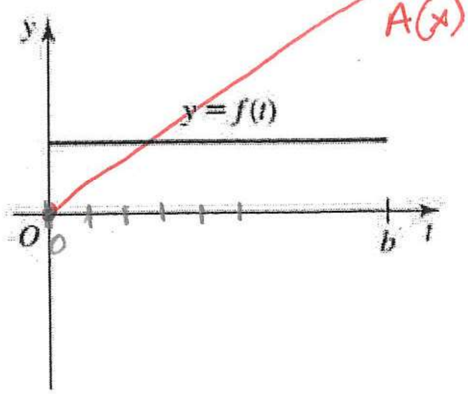
$$= -\frac{d}{dx}\int_4^x \left(t^3 + 5\right)dt$$

$$= -f(x)$$

$$= -\left(x^3 + 5\right) = -x^3 - 5$$

expl 5a: Analyze the graph of $f(t)$ to the right to draw a possible graph of $A(x)=\int_0^x f(t)dt$ on top of this graph.

$A(0) = \int_0^0 f(t)dt = 0$

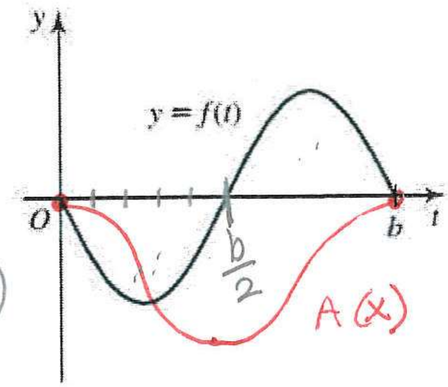


expl 5b: Analyze the graph of $f(t)$ to the right to draw a possible graph of $A(x)=\int_0^x f(t)dt$ on top of this graph.

$A(0) = \int_0^0 f(t)dt = 0$

$A(b) = 0$

$A(b/2)$ is a min (and negative)



expl 6: Use FTC, Part 1, to find the function f that satisfies the equation below. Verify the result by substitution into the equation.

$$\int_0^x f(t)dt = 4\tan x + 5x^2 \cancel{-4}$$

Remember

$$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$

$$f(x) = \frac{d}{dx}\int_0^x f(t)dt = \frac{d}{dx}\left(4\tan x + 5x^2\right)$$

$$= 4\sec^2 x + 10x$$

$$\boxed{\text{So, } f(x) = 4\sec^2 x + 10x}$$

$$f(t) = 4\sec^2 t + 10t.$$

Verify:

$$\int_0^x (4\sec^2 t + 10t)dt \stackrel{?}{=} 4\tan x + 5x^2$$

$$4\tan t + \frac{10t^2}{2}\Big|_0^x \stackrel{?}{=} 4\tan x + 5x^2$$

$$4\tan t + 5t^2\Big|_0^x \stackrel{?}{=} 4\tan x + 5x^2$$

$$4\tan x + 5x^2 - (4\tan 0 + 5\cdot 0^2) \stackrel{?}{=} 4\tan x + 5x^2$$

$$4\tan x + 5x^2 \stackrel{?}{=} 4\tan x + 5x^2 \checkmark$$

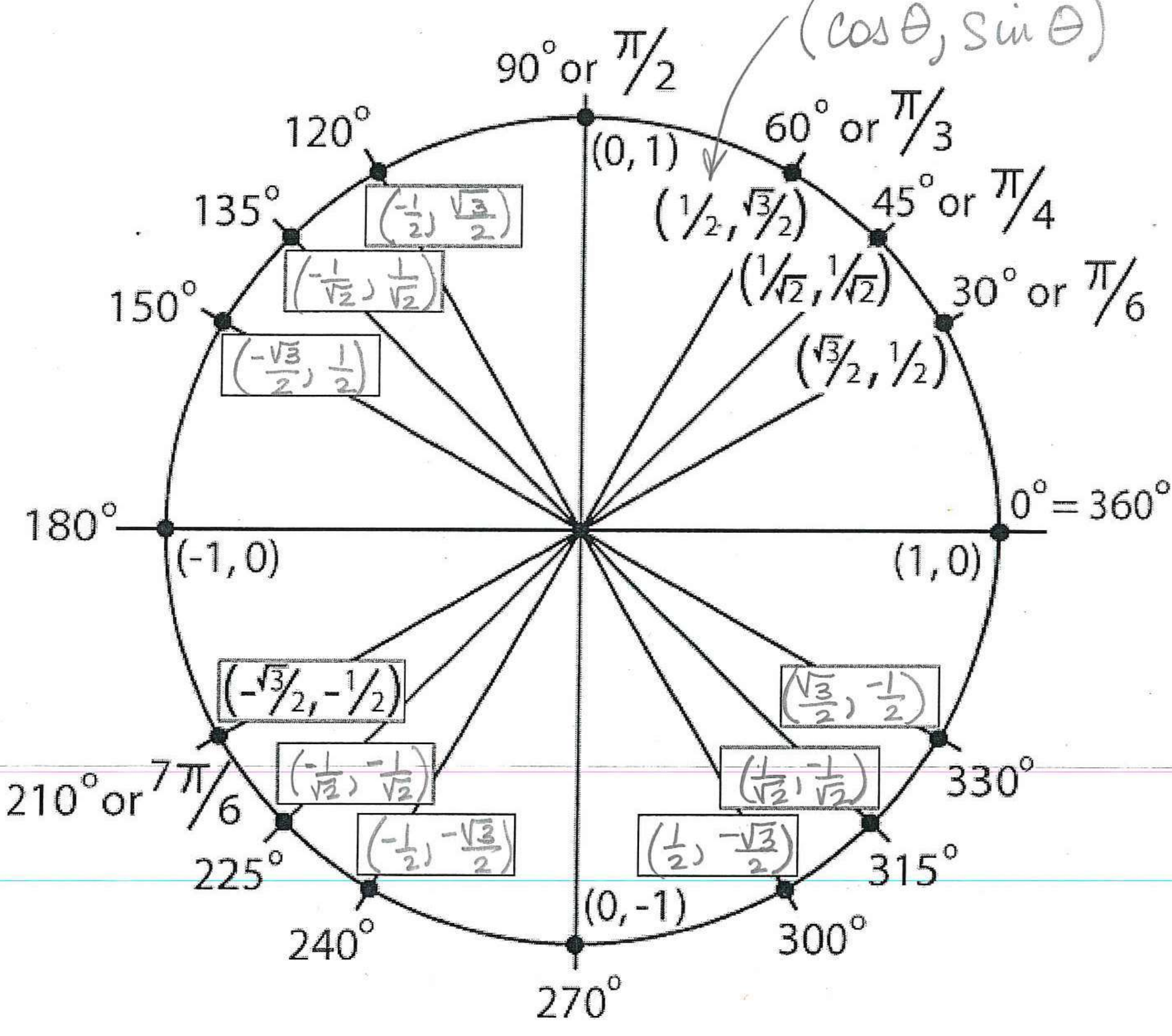
12:00

for 5.3

Unit Circle Trigonometry   NAME:

Below is a unit circle, meaning it is centered on the origin and has a radius of 1. The major angles and their corresponding points, in ordered pair notation, are drawn in and labeled.

Use the given ordered pairs and symmetry about the axes to determine the other ordered pairs. These ordered pairs will then help us find the sine, cosine, and tangent measurements of these angles.



Use your unit circle to complete the following table. Remember the sine, cosine, and tangent of an angle can be found by looking at the corresponding point on the unit circle.

θ in degrees	θ in radians	Point's coordinates (a, b)	Sine of θ $= b$	Cosine of θ $= a$	Tangent of θ $= b/a$
0	$0 * \left(\pi/180\right) = 0$	(1, 0)	0	1	0
30	$30 * \left(\pi/180\right) = \pi/6 \approx .52$	$\left(\sqrt{3}/2, 1/2\right)$	$1/2$	$\sqrt{3}/2$	$1/\sqrt{3}$
45	$45 * \left(\pi/180\right) = \pi/4 \approx .79$	$\left(1/\sqrt{2}, 1/\sqrt{2}\right)$	$1/\sqrt{2}$	$1/\sqrt{2}$	1
60	$60 * \left(\pi/180\right) = \pi/3 \approx 1.05$	$\left(1/2, \sqrt{3}/2\right)$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
90	$90 * \left(\pi/180\right) = \pi/2 \approx 1.57$	(0, 1)	1	0	undefined
120					
135					
150					
180		(−1, 0)			
210	$210 * \left(\pi/180\right) = 7\pi/6 \approx 3.66$	$\left(-\sqrt{3}/2, -1/2\right)$			
225					
240					
270		(0, −1)			
300					
315					
330					
360		(1, 0)			