2:00

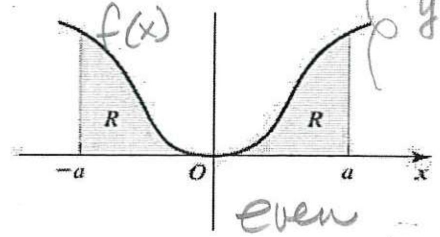
How can knowing a function is even or odd help you find certain integrals? We will see that we can also *average* functions.

Calculus I
Class notes
Working With Integrals: Even or Odd Functions and Average Values (section 5.4)

Do you remember even and odd functions? They are based on useful symmetries. We will see them in specific definite integrals.

We will also explore the average value of a function. It is used to define a **Mean Value Theorem for Integrals** which is similar to the one we saw earlier for derivatives.

Let's start with even and odd functions. Do you recognize this symmetry shown to the right? Is the function said to be even or odd?



Consider $y = x^2$
symm about y-axis

Recall, that the following is true.
If $f(-x) = -f(x)$, then the function is **odd**.
If $f(-x) = f(x)$, then the function is **even**.

So, what does this have to do with definite integrals? Notice the shaded region under the function (call it $f(x)$) between $-a$ and a. What definite integral is equal to this area?

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

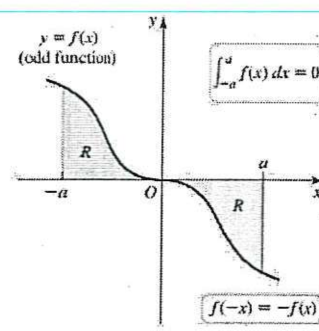
But, of course, this region is symmetric. So, it might occur to you that you need only find the integral for half this region and then double it. Here is what the book has to say.

> **THEOREM 5.4 Integrals of Even and Odd Functions**
> Let a be a positive real number and let f be an integrable function on the interval $[-a, a]$.
> - If f is even, $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.
> - If f is odd, $\int_{-a}^{a} f(x)\,dx = 0$.

Notice, they also have a rule for odd functions.

Do you see why it works? Here is a picture of an odd function to help.

Okay, let's put these to use.



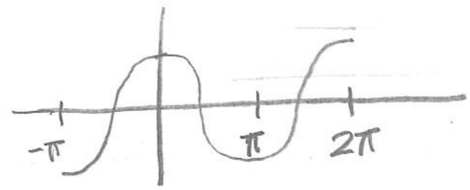
(b)

odd: Symm about origin.
(If you rotate it 180° about the origin, it lies upon itself.)

2:00 5.4

Quickly sketch $y = \cos x$.
Is it even or odd?

expl 1: Use a rule to help find $\int_{-\pi/4}^{\pi/4} \cos x \, dx$.



$$= 2\int_0^{\pi/4} \cos x \, dx$$

$$= 2\left(\sin x \Big|_0^{\pi/4}\right)$$

$$= 2\left(\sin(\pi/4) - \sin(0)\right)$$

$$= 2\left(\frac{1}{\sqrt{2}} - 0\right) = \frac{2}{\sqrt{2}}$$

OR rationaliz $\frac{2\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$

This one is a little wacky. Notice $f(x)$ is neither even *nor* odd. But break it up into two functions ...

expl 2: Use the rules to help find $\int_{-2}^{2} \left(x^9 - 3x^5 + 2x^2 - 10\right) dx$.

$$= \int_{-2}^{2} \underbrace{(x^9 - 3x^5)}_{(odd)} dx + \int_{-2}^{2} \underbrace{(2x^2 - 10)}_{even} dx$$

$$= 0 + 2\int_0^2 (2x^2 - 10) dx$$

$$= 2\left(\frac{2x^3}{3} - 10x\right)\Big|_0^2$$

$$= 2\left(\underbrace{\frac{2 \cdot 2^3}{3} - 10 \cdot 2}_{\text{evaluated at } x=2} - \underbrace{0}_{\text{evaluated } x=0}\right)$$

$$= 2\left(\frac{16}{3} - 20\right)$$

$$= 2\left(\frac{16}{3} - \frac{60}{3}\right) = 2\left(-\frac{44}{3}\right) = \frac{-88}{3}$$

$f(x) = x^9 - 3x^5 + 2x^2 - 10$

Is fnc even or odd?

$f(-x) = (-x)^9 - 3(-x)^5 + 2(-x)^2 - 10$

$f(-x) = -x^9 + 3x^5 + 2x^2 - 10$

Is this equal to $-f(x)$? – No, so $f(x)$ is not odd.

Is this equal to $f(x)$? – No, so $f(x)$ is not even.

$(-x)^9 = (-x)(-x)(-x)(-x)(-x)(-x)(-x)(-x)(-x)$

$(-x)^9 = -x^9$

$(-x)^2 = (-x)(-x)$

$= x^2$

The Average Value of a Function:

We can average five exam scores by adding them up and dividing by 5, right?

We can actually do the same thing with function values (or the y-values of a function). If we consider our partition (into n subintervals) and call the x-values along the way by our familiar x_k^* notation, we would write this average as $\dfrac{f(x_1^*)+f(x_2^*)+f(x_3^*)+...+f(x_n^*)}{n}$.

Do you remember the formula for Δx which involves this same n? Solve it for n and substitute that in the average above and simplify. Rewrite the top in summation notation.

$$\Delta x = \frac{b-a}{n} \rightarrow n = \frac{b-a}{\Delta x} \quad avg = \frac{\sum_1^n f(x_k^*)}{\frac{b-a}{\Delta x}} = \frac{\sum_1^n f(x_k^*)\,\Delta x}{b-a}$$

Lastly, let's take the limit of this as n approaches infinity (or as we consider the number of rectangles getting bigger and bigger). Do you remember what we call this limit of our Riemann Sum? We should end up with what the book has here. integral

DEFINITION Average Value of a Function

The average value of an integrable function f on the interval $[a, b]$ is

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

We pronounce this "f-bar".

$\frac{1}{x} = x^{-1}$

expl 3: Find the average value of $f(x)=\dfrac{1}{x}$ on the interval $[1, e]$. Give both exact and rounded (three decimal places) forms.

$$\int_a^b \frac{1}{x}\,dx = \ln|x|\Big|_a^b$$

$$\bar{f} = \frac{1}{e-1}\int_1^e \frac{1}{x}\,dx = \frac{1}{e-1}\ln|x|\Big|_1^e$$

$$= \frac{1}{e-1}(\ln|e| - \ln|1|)$$

$$= \frac{1}{e-1}(\ln e - \ln 1) = \frac{1}{e-1}(\log_e e - \log_e 1)$$

$e^? = 1$

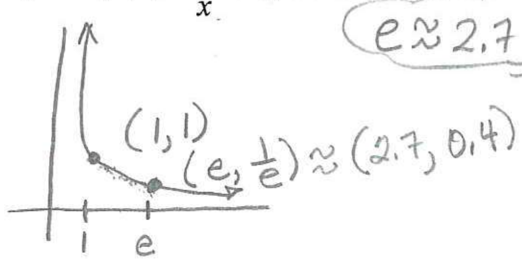
$$= \frac{1}{e-1}(1-0) = \frac{1}{e-1}\cdot 1$$

So, $\boxed{\bar{f} = \frac{1}{e-1}}$ exact

$\boxed{\bar{f} \approx 0.582}$ rounded

Sketch a graph of $f(x)=\dfrac{1}{x}$ to get a sense of what we found.

$e \approx 2.7$



quadratic function

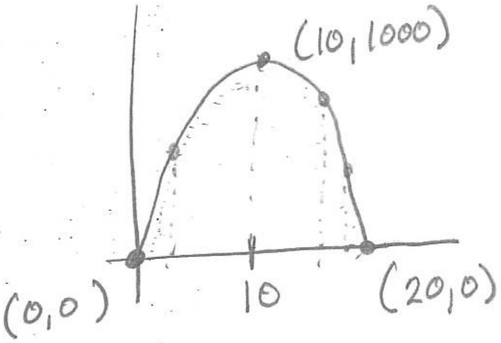
$y = ax^2 + bx + c$

expl 4: Find the average distance between the x-axis and the parabola $f(x) = 10x(20 - x)$ on the interval [0, 20].

We could use the distance formula but it is *not* necessary. Look at the graph of $f(x)$. How does this distance show up on the graph?

just the $f(x)$ values

$f(x) = 200x - 10x^2$



$[-10, 25] \times [-10, 1200]$

x-value of the vertex

$x = -b/2a$

$= -200/2(-10)$

$x = 10$

Simplify the integrand first.

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{20-0}\int_0^{20} (200x - 10x^2)\,dx$$

$$= \frac{1}{20}\left(\frac{200x^2}{2} - \frac{10x^3}{3}\right)\Big|_0^{20}$$

$$= \frac{1}{20}\left(\frac{200 \cdot 20^2}{2} - \frac{10 \cdot 20^3}{3} - 0\right)$$

$$= \frac{1}{20}\left(\frac{\overset{100}{\cancel{200}} \cdot 400}{\cancel{2}} - \frac{10 \cdot 8000}{3}\right)$$

$$\bar{f} = \frac{2000}{3} \approx 666.67$$

✓

section 4.2

The Mean Value Theorem for Integrals:
We have seen the Mean Value Theorem for derivatives earlier. This ensured that there would be some x-value, call it c, in a given interval (a, b) whose derivative was equal to the slope of the tangent line between the interval's endpoints. Can you write the MVT formula here?

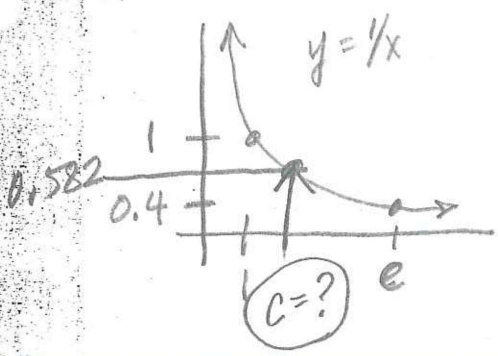
$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

This theorem and the Fundamental Theorem of Calculus lead us to the following conclusion about the average value of a function. See the book for its proof.

> **THEOREM 5.5 Mean Value Theorem for Integrals** x-value
> Let f be continuous on the interval $[a, b]$. There exists a point c in (a, b) such that
> $$f(c) = \bar{f} = \frac{1}{b-a}\int_a^b f(t)\, dt.$$

expl 5: Find the point guaranteed to exist whose function value is equal to the average value of $f(x) = \dfrac{1}{x}$ on the interval $[1, e]$. Leave your answer in exact form.

Use the exact form of this average value found back in example 3.



$$f(c) = \bar{f}$$

$$\frac{1}{c} = \frac{1}{e-1}$$

$$e-1 = c$$

$$c = e-1 \quad \text{or} \quad c \approx 1.72$$

So, we found $c=e-1$ is the number such that $f(c) = \bar{f}$.