2:00

At first glance, $\int 5x^4\left(x^5+3\right)^4 dx$ looks ghastly. But do you notice anything kinda weird about $5x^4$ and x^5+3?

der of that

Calculus I
Class notes
Integrals with u-Substitution (section 5.5)

To find $\int 5x^4\left(x^5+3\right)^4 dx$ by methods we know, we would need quite a bit of algebra to expand it (and then take the integral term-by-term). We could but it would be tedious and we're liable to make mistakes along the way. There is an easier way. Our new method will be useful for a surprising number of integrals we see in calculus.

The first thing to notice is that $\frac{d}{dx}\left(x^5+3\right)=5x^4$ as was teased above.

The second thing we need to know is more subtle. Let's start with a definition of a function $u=x^5+3$. Here, u is said to be a function in x. This means that u is *not* an ordinary variable; it is a *function* with x as its independent variable. Now, what is the derivative of u? Write it as du.

Do you remember the Chain Rule?

$u = x^5+3$

$du = 5x^4\,dx$

$\frac{du}{dx} = 5x^4 \rightarrow du = 5x^4 dx$

Let's rewrite $\int 5x^4\left(x^5+3\right)^4 dx$ with this new function u and its derivative in place.

du $\rightarrow \int u^4 du$

u^4

Holy integration, Batman! Did we just create an integral for which we already know its rule? Boom, shakalaka! Find this integral but do *not* forget that we started with x, *not* u.

$\int u^4 du$

$= \frac{1}{5}u^5 + C = \boxed{\frac{1}{5}\left(x^5+3\right)^5 + C} \quad (C \in \mathbb{R})$

We are back to indefinite integrals so do *not* forget the constant at the end.

Here is how the book lays this out.

THEOREM 5.6 Substitution Rule for Indefinite Integrals
Let $u = g(x)$, where g is differentiable on an interval, and let f be continuous on the corresponding range of g. On that interval,

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

In practice, Theorem 5.6 is applied using the following procedure.

PROCEDURE Substitution Rule (Change of Variables)

1. Given an indefinite integral involving a composite function $f(g(x))$, identify an inner function $u = g(x)$ such that a constant multiple of $g'(x)$ appears in the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)\,dx$ in the integral.
3. Evaluate the new indefinite integral with respect to u.
4. Write the result in terms of x using $u = g(x)$.

Disclaimer: Not all integrals yield to the Substitution Rule.

But they do all yield to a good whacking with a solid stick.

expl 1: Work your magic. Let $u = \sin x$

$$\int (\underbrace{\sin^3 x}_{u^3} \cdot \underbrace{\cos x)\,dx}_{du} \qquad du = \cos x\,dx$$

$$= \int u^3\,du$$

$$= \frac{1}{4}u^4 + c = \boxed{\frac{1}{4}(\sin^4 x) + c,\ c \in \mathbb{R}}$$

expl 2: What, you say you want another?

$$\int 2x\left(x^2+1\right)^4 dx$$

du

Let $u = x^2+1$

$du = 2x\,dx$

$= \int u^4\,du$

$= \frac{1}{5}u^5 + c = \boxed{\frac{1}{5}(x^2+1)^5 + c,\ c \in \mathbb{R}}$

expl 3: Do this one. Let $u = 10x + 3$. But then what is du and how do we use that? (Hint: You'll need to solve for dx.)

$$\int \frac{1}{10x+3}\,dx$$

$\int \frac{1}{x}\,dx = \ln|x| + c$

$u = 10x + 3$

$du = 10\,dx$

$\frac{du}{10} = dx \rightarrow \frac{1}{10}\,du = dx$

$= \frac{1}{10}\int \frac{1}{u}\,du$

$= \frac{1}{10}\left(\ln|u| + c\right) = \frac{1}{10}\ln|u| + c$ because $\frac{1}{10}c$ is still a constant.

$= \frac{1}{10}\ln|10x+3| + c,\ c \in \mathbb{R}$

Rules, Rules, Rules:

There are many rules that people have discovered over the years for various integrals. Some need to be committed to memory. For many others, you need to simply know that there is likely to be a rule somewhere that can be looked up. Here are a few.

Table 5.6 General Integration Formulas

1. $\int \cos ax\, dx = \frac{1}{a} \sin ax + C$
2. $\int \sin ax\, dx = -\frac{1}{a} \cos ax + C$
3. $\int \sec^2 ax\, dx = \frac{1}{a} \tan ax + C$
4. $\int \csc^2 ax\, dx = -\frac{1}{a} \cot ax + C$
5. $\int \sec ax \tan ax\, dx = \frac{1}{a} \sec ax + C$
6. $\int \csc ax \cot ax = -\frac{1}{a} \csc ax + C$

Here is this same table from the previous edition of the book. Notice it has some other rules; I wanted to play with them too. In all of the rules, a stands for some real number.

Table 5.6 General Integration Formulas

1. $\int \cos ax\, dx = \frac{1}{a} \sin ax + C$
2. $\int \sin ax\, dx = -\frac{1}{a} \cos ax + C$
3. $\int \sec^2 ax\, dx = \frac{1}{a} \tan ax + C$
4. $\int \csc^2 ax\, dx = -\frac{1}{a} \cot ax + C$
5. $\int \sec ax \tan ax\, dx = \frac{1}{a} \sec ax + C$
6. $\int \csc ax \cot ax = -\frac{1}{a} \csc ax + C$
7. $\int e^{ax}\, dx = \frac{1}{a} e^{ax} + C$
8. $\int b^x\, dx = \frac{1}{\ln b} b^x + C,\ b > 0,\ b \neq 1$
9. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$
10. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C,\ a > 0$
11. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \left|\frac{x}{a}\right| + C,\ a > 0$

For numbers 9-11, the order of the integrand terms matter.

As you get into calculus more, you will find that there are lots of people online (like our lovely Paul Dawkins) who have posted pages and pages and pages (and pages, oh lordy loo!) of rules.

We will explore problems that use these. Your MML homework will likely stick to problems that use the rules given at the top of the page.

Rule #10: $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}(x/a) + C,\ a>0$

expl 4: Use a rule. Think it's cool. You don't rhyme? Give it time.

$$\int \frac{1}{\sqrt{1-9x^2}}\,dx$$

$a=1$

$u=3x$

$du = 3dx$

$\frac{1}{3}du = dx$

Find the closest rule you can and write it here. Define u so that it works.

$$= \frac{1}{3}\int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \frac{1}{3}\left(\sin^{-1}(u/1) + c\right)$$

$$= \frac{1}{3}\sin^{-1}(u) + c = \boxed{\frac{1}{3}\sin^{-1}(3x) + c,\ c\in\mathbb{R}}$$

expl 5: Here, we use y as our independent variable to mix it up a bit. Let $u = y+1$.

$$\int \frac{y^2}{(y+1)^4}\,dy$$

$u = y+1$

$du = 1dy$

But that is *not* all you have to do. If $u = y+1$, then what is y?

You'll need more algebra. FOIL on top and distribute bottom…

$y = u-1$

$y^2 = (u-1)^2$

$= (u-1)(u-1)$

$= u^2-2u+1$

$$= \int \frac{u^2-2u+1}{u^4}\,du$$

$$= \int \left(\frac{u^2}{u^4} - \frac{2u}{u^4} + \frac{1}{u^4}\right) du$$

$$= \int \left(\frac{1}{u^2} - \frac{2}{u^3} + \frac{1}{u^4}\right) du$$

$$= \int \left(u^{-2} - 2u^{-3} + u^{-4}\right) du$$

Power Rule

$$= \frac{u^{-1}}{-1} - \frac{2u^{-2}}{-2} + \frac{u^{-3}}{-3} + C = -\frac{1}{u} + \frac{1}{u^2} - \frac{3}{u^3} + C$$

$$= \boxed{-\frac{1}{y+1} + \frac{1}{(y+1)^2} - \frac{3}{(y+1)^3} + C,\ C\in\mathbb{R}}$$

Integration takes equal amounts of luck and creativity.

expl 6: Take what you learned on the last example and step it up a notch. If we let $u = 3x + 2$, then what is $x + 1$?

$\int (x+1)\sqrt{3x+2}\,dx$

?

$u = 3x+2$
$du = 3dx$
$\frac{1}{3}du = dx$

Solve for x
$u = 3x+2$
$\frac{u-2}{3} = x$
So, $x+1 = \frac{u-2}{3} + 1$
$= \frac{u-2}{3} + \frac{3}{3}$
$x+1 = \frac{u+1}{3}$
$= \frac{1}{3}(u+1)$

$\frac{1}{3}\int (x+1)\sqrt{u}\,du$

$= \frac{1}{3}\frac{1}{3}\int (u+1)u^{1/2}\,du$

$= \frac{1}{9}\int (u^{3/2} + u^{1/2})\,du$

$= \frac{1}{9}\left(\frac{2u^{5/2}}{5} + \frac{2u^{3/2}}{3} + c\right)$

Dividing by a fraction means "flip and multiply".

$$\boxed{= \frac{2}{45}(3x+2)^{5/2} + \frac{2}{27}(3x+2)^{3/2} + C,\ C \in \mathbb{R}}$$

The Return of Definite Integrals:

indefinite

We started this journey by finding $\int 5x^4\left(x^5+3\right)^4 dx$. What if we needed $\int_0^2 5x^4\left(x^5+3\right)^4 dx$?

We have a method for evaluating definite integrals. So, we should be able to use our u-substitution to find this integral *without* the limits of integration (again, this is just an antiderivative) and then use the FUN-damental Theorem of Calculus to finish it out to find the integral from 0 to 2. Right?

There is a small problem. Actually, it can be a *big* problem if you do *not* mind your p's and q's (as my high school algebra teacher loved saying).

Mind your x's and u's?

It would *not be right* to write $\int_0^2 5x^4\left(x^5+3\right)^4 dx = \int_0^2 u^4\,du$ using our substitution $u = x^5 + 3$. Why?

x values

these are x values, not u.

The Return of Definite Integrals (Continued):

You could just drop the limits of integration from the integrals involving u and then put them back when you get back to x. Or, you can find u when x is 0 and when x is 2 and use those as the limits of the integral involving u.

The book puts it this way.

THEOREM 5.7 Substitution Rule for Definite Integrals
Let $u = g(x)$, where g' is continuous on $[a, b]$, and let f be continuous on the range of g. Then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du. \qquad (2)$$

Find u when $x=a$ and $x=b$.

expl 7: Use the book's method to find $\int_0^2 5x^4(x^5+3)^4\,dx$.

$u = x^5+3$

$du = 5x^4\,dx$

Find u when $x=0$

$u = 0^5+3 = 3$

Find u when $x=2$

$u = 2^5+3 = 35$

$$= \int_3^{35} u^4\,du$$

$$= \frac{1}{5}u^5 \Big|_3^{35}$$

$$= \frac{1}{5}(35^5-3^5) = \boxed{10{,}504{,}326.4}$$

expl 8: Find the following.

$$\int_0^{\pi/4} \frac{\sin x}{\cos^3 x}\,dx$$

It is tempting to let $u = \cos^3 x$. Find du to see why this would *not* be a good idea.

$\cancel{du = 3\cos^2 x(-\sin x)}$

$\cancel{du = -3\sin x\cos^2 x}$

oops - does not work!

Let $u = \cos x$

$du = -\sin x\,dx$

$-du = \sin x\,dx$

Find u when $x=0$

$u = \cos 0 = 1$

Find u when $x = \pi/4$

$u = \cos(\pi/4) = 1/\sqrt{2}$

$$= -1\int_1^{1/\sqrt{2}} \frac{1}{u^3}\,du$$

$$= -1\int_1^{1/\sqrt{2}} u^{-3}\,du$$

$$= -1\left(\frac{u^{-2}}{-2}\right)\Big|_1^{1/\sqrt{2}}$$

$$= \frac{1}{2u^2}\Big|_1^{1/\sqrt{2}} = \frac{1}{2(1/\sqrt{2})^2} - \frac{1}{2(1)^2} = \frac{1}{2/2} - \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2}$$

Rule #3 (pg 4 top) $\int \sec^2(ax)\,dx = \frac{1}{a}\tan(ax) + c$

expl 9: Seek out a special rule for this one. Check your answer.

$\int \sec^2(10x)\,dx$ $\quad a=10$

$= \boxed{\frac{1}{10}\tan(10x) + c}$

expl 10: For this last one, start with a u-substitution. However, the rule you need to complete the integral is *not* in these notes. Look through the Paul Dawkins handout (given out in section 5.3) to find the formula.

$\int \frac{e^{2x}}{3e^{2x}+1}\,dx$

Let $u = e^{2x}$

$du = 2e^{2x}\,dx$

$\frac{1}{2}du = e^{2x}\,dx$

$\frac{d}{dx}(e^x) = e^x$

$\frac{d}{dx}(e^{g(x)}) = e^{g(x)} \cdot g'(x)$

Chain Rule

$= \frac{1}{2}\int \frac{du}{3u+1}$

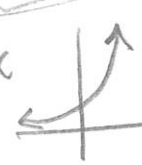
$a = 3$
$b = 1$

Paul Dawkins

$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$

$= \frac{1}{2}\frac{1}{3}\ln|3u+1| + c, \; c \in \mathbb{R}$

$= \boxed{\frac{1}{6}\ln|3e^{2x}+1| + c, \; c \in \mathbb{R}}$

$y = e^x$  always positive

OR

$(e^x)^2 = e^{2x}$

$= \boxed{\frac{1}{6}\ln(3e^{2x}+1) + c, \; c \in \mathbb{R}}$

How can the absolute value of something that is *always* positive be rewritten?