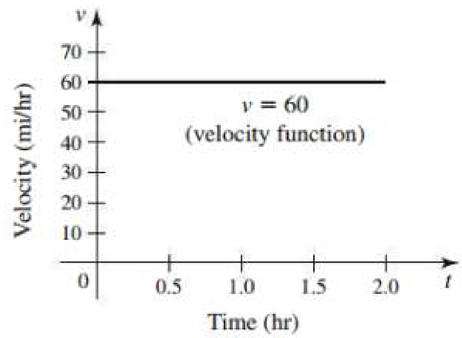
Chapter 5 tells us more about integrals. It still amazes me how we see them in the graph of a function.

Calculus I
Class notes
Riemann Sums: Approximating Areas Under Curves (section 5.1)

So, we know that $f'(x)$ is the rate of change of a function shown as the slope of the tangent line at some x-value. That was pretty cool but it gets even cooler when we investigate the graphical meaning of $\int f(x)dx$.

For our first exploration, consider a car moving along a straight road at a velocity of 60 mph. This is a constant velocity so its graph is a horizontal line, isn't it? Here's a picture of the first two hours of the drive.

How far does it travel in this time? (Distance would be $s(t) = \int v(t)dt$.)



Recall, we can label the distance traveled the car's **displacement**. Now, and this is the truly mind-blowing part, draw in the right side of the rectangle this function makes with the axes and find the area of this rectangle.

How is this area related to the total distance driven?

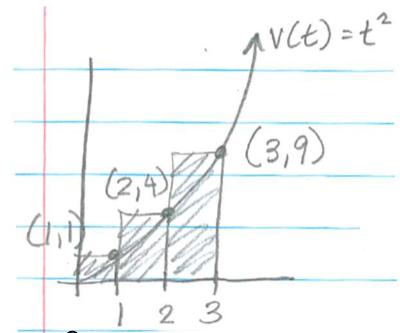
Do you think it is a coincidence that this area equals the total distance driven? Life is full of coincidences, after all. "No", you say. You say, "Life is full of coincidences, but it is also full of beautiful math that explains at least some of those coincidences." You would be right. Well done. Let's get more creative with this velocity function.

Let's say our car's velocity function is $v(t) = t^2$ where t is the number of hours. Give yourself a quick graph but remember that we will only need the right side of the parabola. (Why?)

Label the t-values 1, 2, and 3. Consider the *distance driven* in these first three hours.

How would we find the area under this curve for [0, 3]? It's all curvy on top so it's *not* so easy as before. However, we could *approximate* the area by using a series of rectangles.

Here is my drawing of $v(t) = t^2, t \geq 0$ with the three points at $t = 1$, 2, and 3 marked. From those points, I drew and shaded rectangles. Calculate this shaded area.



How well do these rectangles approximate the actual area?

Did we overestimate or underestimate the actual area?

This is called a **regular partition** of the interval [0, 3].

Try this one. Notice how I chose to draw the rectangles on this second attempt. Since, we are going to $t = 3$, there look to be only two rectangles. (The first rectangle technically has *no* height.) Find this area.

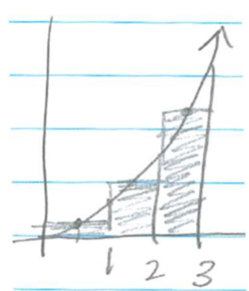
How good is this approximation? Would you say that it overestimated or underestimated the actual area?

We do this work by dividing the interval [0, 3] into three subintervals, [0, 1], [1, 2], and [2, 3]. When we used the *right* endpoint of each interval to draw the rectangles, we *overestimated* the actual area. When we used the *left* endpoints, we *underestimated* the area.

Let's cut the difference and center the rectangles on the intervals themselves. To find this area, you will need to find some $v(t)$ values first.

Complete the table on the far right and then find this area. You should see that it gives a value somewhere in the middle of the first two estimates.

More importantly, look at the picture and decide if the estimate is a good one. Does it *really* look like it captured the area under the curve?



t	$\boldsymbol{v(t) = t^2}$
0.5	
1.5	
2.5	

Our Old Friend Limit Lends a Hand:
Imagine if we had used more rectangles. We could have pinpointed the area even better with more, skinnier rectangles. The more rectangles we used, the better our approximation would become.

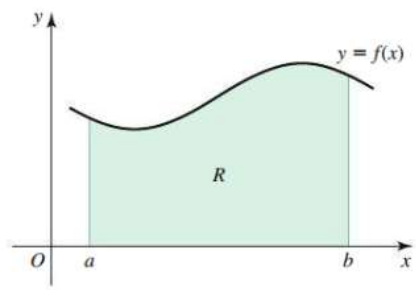
What if we had an infinite number of rectangles? You say we can't do that? Okay, but we can *imagine* it, right?

To find this area exactly, let n be the number of rectangles we subdivide the area into. We will take the limit, as $n \to \infty$, of the sum of these rectangles' areas. This, we will see as we go through the chapter, is the *exact* area under the curve. For the example we started with, this would be the *exact* distance the car had traveled. Let's do some work to get there.

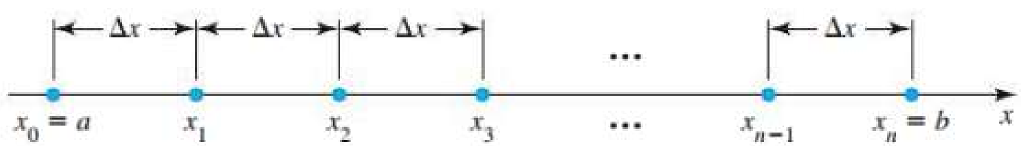
Regular Partitions:

Consider a function $f(x)$ that is continuous and non-negative on the interval $[a, b]$. We will label the region whose area we seek R.

Now, divide the interval $[a, b]$ into n subintervals of equal length. To define these subintervals, we will assign intermediate x-values from x_0 (which is a) to x_n (which is b).



They are equally spaced and we call the distance between two successive values our familiar Δx. Here's a picture of that followed by the definition of a **regular partition**.



DEFINITION Regular Partition
Suppose $[a, b]$ is a closed interval containing n subintervals

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$$

of equal length $\Delta x = \dfrac{b-a}{n}$, with $a = x_0$ and $b = x_n$. The endpoints $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ of the subintervals are called **grid points**, and they create a **regular partition** of the interval $[a, b]$. In general, the kth grid point is

$$x_k = a + k\Delta x, \text{ for } k = 0, 1, 2, \ldots, n.$$

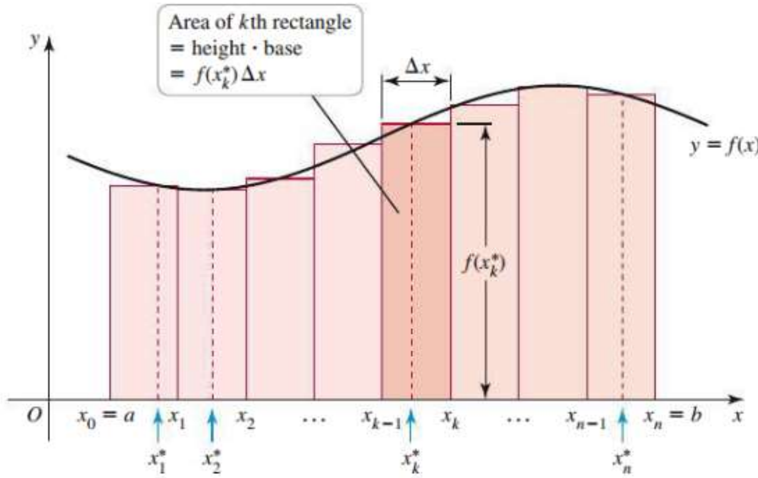
Give yourself a moment to verify their formula for Δx and x_k.

Again, imagine the area we are after as subdivided into many rectangles. In every one of these rectangles, we will choose some x-value to find its area.

We will call that x-value x_k^*. This is how the book shows it.

Would you agree with how they calculate the area of this rectangle?

Take a moment to understand why they are using $f(x_k^*)$.



Riemann Sums:
Wikipedia.org tells me that Riemann Sums are named after the nineteenth century German mathematician **Bernhard Riemann**. One can only assume he was teased mercilessly for his name until he showed them all by becoming a great mathematician. Bet they got nothing named after them.

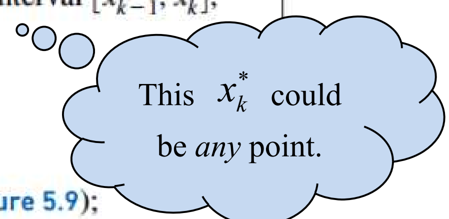
DEFINITION Riemann Sum

Suppose f is defined on a closed interval $[a, b]$, which is divided into n subintervals of equal length Δx. If x_k^* is any point in the kth subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$, then

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

is called a **Riemann sum** for f on $[a, b]$. This sum is called

- a **left Riemann sum** if x_k^* is the left endpoint of $[x_{k-1}, x_k]$ (Figure 5.9);
- a **right Riemann sum** if x_k^* is the right endpoint of $[x_{k-1}, x_k]$ (Figure 5.10); and
- a **midpoint Riemann sum** if x_k^* is the midpoint of $[x_{k-1}, x_k]$ (Figure 5.11), for $k = 1, 2, \ldots, n$.



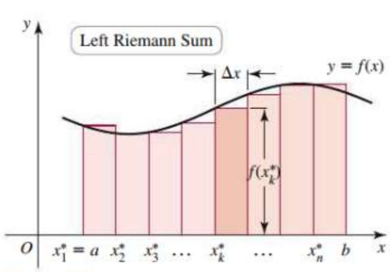


Figure 5.9

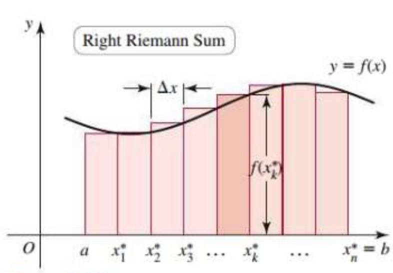


Figure 5.10

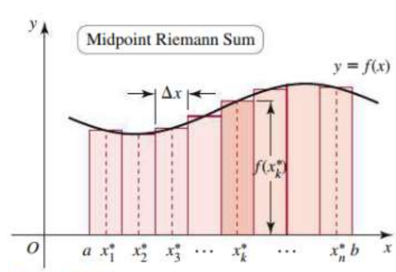


Figure 5.11

Summation (Sigma) Notation:
Writing the calculations for these areas is repetitive and hard on the hand. Mathematicians, being the lazy buggers they are, came up with shorthand notation. We will use the Greek capital letter Σ (sigma).

If we want to write 1 + 2 + 3 + … + 10, we will abbreviate that as $\sum_{k=1}^{10} k$. The **index** k takes on every integer value from 1 through 10. This is an arbitrary symbol (a dummy variable) and you could use n or m as easily as k. The **summand** is the name for the expression that follows the sigma (and is the thing you add again and again as the index changes from the **lower limit** to the **upper limit**). In this case, the summand is k but it could be any expression. We will play with this notation a bit but mainly use it for our Riemann Sums.

expl 1: Write out the following in expanded form. Do *not* bother finding the sum.

$$\sum_{k=1}^{5} k^2$$

There are many rules; here are only sum of them.

Constant Multiple Rule $\displaystyle\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k.$

Addition Rule $\displaystyle\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k.$

expl 2: Use one of the rules above to rewrite the following. Then expand the form.

$$\sum_{k=1}^{3} 5x^k$$

expl 3: Use summation notation to rewrite the Riemann Sum given on the previous page.

These formulas have been known and used for centuries. Those for higher exponents are complicated.

You may also put these rules to work.

THEOREM 5.1 Sums of Powers of Integers
Let n be a positive integer and c a real number.

$$\sum_{k=1}^{n} c = cn \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Here's some space so you can play with them.

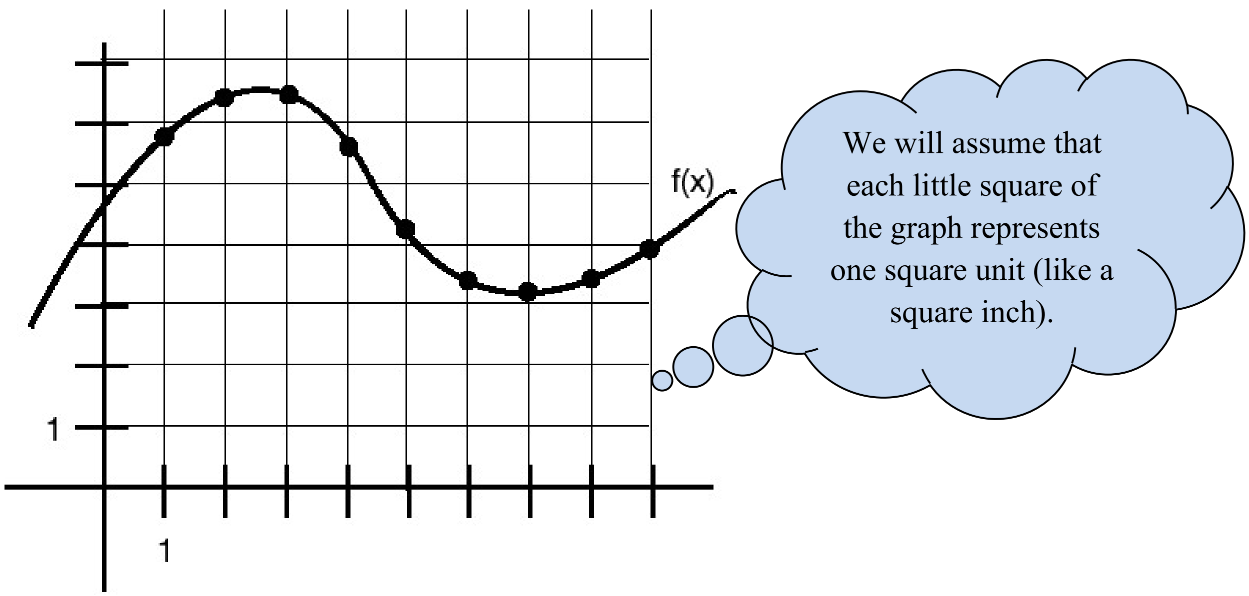
Again, the reason we study this is the Riemann Sum. So, here it is in summation notation. You will also see that we are given the formulas for x_k^*. They differ depending on the type of Riemann Sum you are after.

DEFINITION Left, Right, and Midpoint Riemann Sums in Sigma Notation
Suppose f is defined on a closed interval $[a, b]$, which is divided into n subintervals of equal length Δx. If x_k^* is a point in the kth subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$, then the **Riemann sum** for f on $[a, b]$ is $\sum_{k=1}^{n} f(x_k^*)\Delta x$. Three cases arise in practice.

- $\sum_{k=1}^{n} f(x_k^*)\Delta x$ is a **left Riemann sum** if $x_k^* = a + (k-1)\Delta x$.
- $\sum_{k=1}^{n} f(x_k^*)\Delta x$ is a **right Riemann sum** if $x_k^* = a + k\Delta x$.
- $\sum_{k=1}^{n} f(x_k^*)\Delta x$ is a **midpoint Riemann sum** if $x_k^* = a + (k - \frac{1}{2})\Delta x$.

expl 4: We are interested in estimating the area under the curve $f(x)$ shown below from $x = 0$ to $x = 9$. We will do this by the method of Riemann Sums.

Notice the points $(x, f(x))$ where x is an integer are marked with big dots. **For each one of these dots**, draw a **horizontal** segment over to the vertical line directly to its left. These segments complete rectangles (width equals 1) formed by the positive x-axis and the vertical lines drawn in the graph. Shade in these rectangles. Notice how this shaded area closely resembles the area under the curve.



Estimate each $f(x)$ value and record it on the table below. Round to the nearest quarter unit. Since the rectangles all have a width of 1, this value is also the area of the rectangle.

Rectangle	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
***f*(*x*) or Area**									

What is the total area? Again, notice the area of the rectangles estimates the area under the curve of $f(x)$ [and above the x-axis] from $x = 0$ to $x = 9$.

expl 5: Let $f(x) = \cos x$ on $\left[0, \frac{\pi}{2}\right]$. Use $n = 4$ subintervals.

a.) Sketch the function on the given interval twice. Use up the space provided.

b.) Calculate Δx and the grid points $x_0, x_1, x_2, \ldots x_n$. Label them on your graphs above.

c.) Illustrate the left and right Riemann Sums on your pictures. Label which is which. Also, tell which underestimates and which overestimates the area under the curve. You do *not* need to find these sums.

Do you know why a Riemann Sum would give an underestimate versus an overestimate yet?

expl 6a: Approximate the displacement of the object whose velocity is given below. Do so on the interval $0 \le t \le 4$ by subdividing into $n = 4$ subintervals. Use left endpoints. You may assume the units are meters per second.

$$v = \frac{t+3}{6} = \frac{1}{6}t + \frac{1}{2}$$

Form a graph to help visualize. Find Δt and the grid points. Complete the tables below to help organize.

G*rid point** ***t	***v*(*t*)**

Left endpoint ***t***	**Area of rectangle**

What is the total area? Include units.

We can use the calculator for these repetitive calculations.

TI Calculator Instructions:

1. Start on the home screen. Press the **2nd** and then **STAT** buttons to get to the **LIST** command. Arrow over to **MATH**. Select the option **5: sum(**. [This will be the sigma part of the expression.]

2. Navigate back to the **LIST** command as before. This time arrow over to **OPS**. Select the option **5: seq(**. [This will be how you tell it the lower and upper limits for k.]

3. You will enter the expression you want summed using x, the variable x itself, the lower limit of the summation, and the upper limit of the summation, all separated by commas. Finish with two parentheses at the end and hit **ENTER**.

Newer calculators may have a different interface.

expl 6b: Try this process out to find the summation from the previous page. Write it here in both sigma notation and what it looks like on the calculator.

expl 7: Let $f(x)=\sqrt{x}$ on $[1,3]$. Use $n=4$ subintervals.

a.) Sketch the function on the given interval. Calculate Δx and the grid points $x_0, x_1, x_2, \ldots x_n$. Label them on your graph.

b.) Illustrate the midpoint Riemann Sum on your picture. Find the four values of x_k^* needed to calculate the Riemann Sum by hand. Start with $\sum_{k=1}^{4} f\left(x_k^*\right)\Delta x$ and find this sum using your values. The table will help you organize.

Midpoint Riemann Sum:
$x_k^* = a + \left(k - \frac{1}{2}\right)\Delta x$

x_k^*	$f(x_k^*)$

expl 8a: For the following function, use sigma notation to write out the left and right Riemann Sums. An interval and number of subintervals is given. Then use a calculator to find the sums.

$f(x) = x^3 + 1$; $[0, 2]$; $n = 50$

Left Riemann Sum:

$x_k^* = a + (k-1)\Delta x$

Right Riemann Sum:

$x_k^* = a + k\Delta x$

expl 8b: Taking your sums into account, estimate the area under the curve of $f(x)$ in the given interval. Sketch a quick picture of the region whose area we have estimated.

In the sections to come, we will learn how to find this area exactly.