

Some theorems get all the luck in naming.
This one really does live up to its name. We
connect antiderivatives and definite integrals.

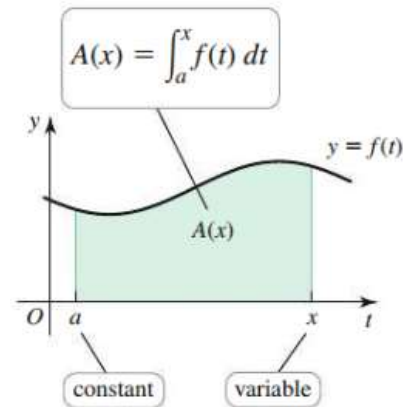
Calculus I
Class notes

Fundamental Theorem of Calculus (section 5.3)

We know that a function's definite integral is the area under the curve, bounded by the x -axis.

We introduce the **area function** which just gives a name to that concept.

It is defined below and here is a picture.



DEFINITION Area Function

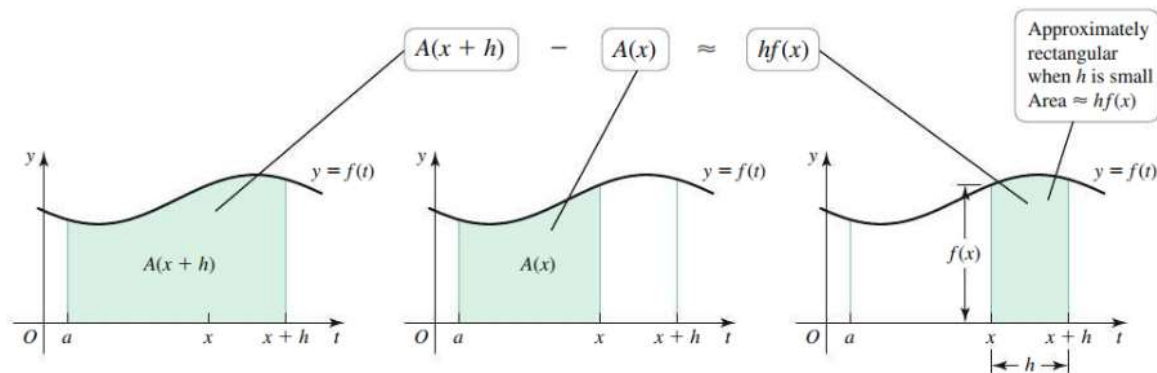
Let f be a continuous function, for $t \geq a$. The **area function for f with left endpoint a** is

$$A(x) = \int_a^x f(t) dt,$$

where $x \geq a$. The area function gives the net area of the region bounded by the graph of f and the t -axis on the interval $[a, x]$.

We will start with a justification of our FUN-damental theorem.

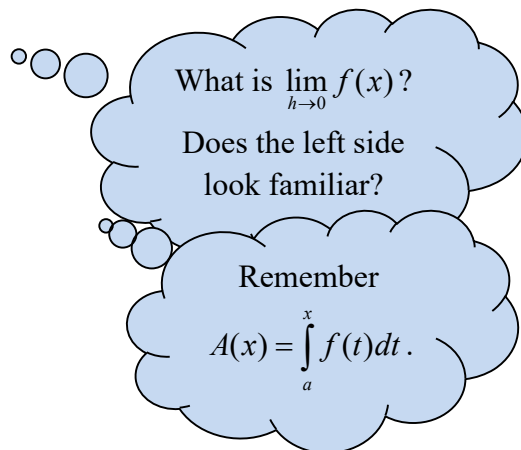
Consider these pictures which use the basic idea that you can subtract one area from another, leaving the leftover part.



We are making the assumption (*not* necessarily evident from the right-most picture) that h is small. In fact, what happens when we take h to approach 0? Does that sound familiar?

We will assume $h > 0$ but the argument is similar where $h < 0$.

We start with the equation $A(x+h) - A(x) \approx h \cdot f(x)$. Divide both sides by h and observe that as h tends toward 0 (how do you write that?), the approximation improves. In fact, we can replace the approximately equal sign with an equal sign.



This argument gives us the first part of the Fundamental Theorem of Calculus.

THEOREM 5.3 (PART 1) Fundamental Theorem of Calculus

If f is continuous on $[a, b]$, then the area function

$$A(x) = \int_a^x f(t) dt, \quad \text{for } a \leq x \leq b,$$

is continuous on $[a, b]$ and differentiable on (a, b) . The area function satisfies $A'(x) = f(x)$. Equivalently,

$$A'(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x),$$

which means that the area function of f is an antiderivative of f on $[a, b]$.

So, this area function is an antiderivative of the function $f(x)$. That is definitely good to know but we are after the main event. What does this tell us about finding definite integrals?

Let's say that we have another antiderivative of f . We'll call this one $F(x)$. Do you remember that any two antiderivatives differ by a constant? How would you write that A and F are two antiderivatives that differ by a constant? Isolate $F(x)$.

We will consider x -values
such that $a \leq x \leq b$.

Find $F(b) - F(a)$.

Do you remember
what $A(a)$ would
have to be?

Lastly, $A(b) = \int_a^b f(x) dx$.

Do you see that this gives us a way to calculate $\int_a^b f(x) dx$? That is the second part of our theorem.

THEOREM 5.3 (PART 2) Fundamental Theorem of Calculus

If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a).$$

Shorthand notation for $F(b) - F(a)$ is $F(x)\big|_a^b$.

So, in short, to find the value of a definite integral, we find the integral (as we saw it before) as an antiderivative. We then evaluate the antiderivative at the limits of integration and subtract (top minus bottom).

Voilà! We have the area under the function's curve, bounded by the x -axis.

We have come full circle back to the antiderivatives where we began.

Let's practice. I will refer to the theorem as FTC.

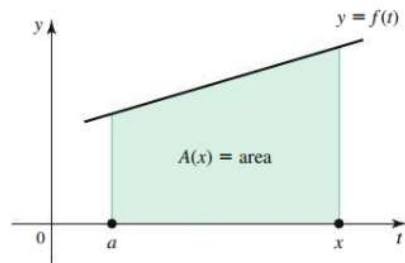
Handout: Paul Dawkins Calculus Cheat Sheet:

Once again, Paul gives us a concise summary of all we know so far and then some.

expl 1: For the function $f(t) = 2t + 5$ shown to the right,

find the area function $A(x) = \int_a^x f(t) dt$. Verify that

$A'(x) = f(x)$. Assume $a = 1$.



The area of a trapezoid is

$$A = \frac{1}{2} h (b_1 + b_2) \text{ where } h$$

is the distance between
parallel bases b_1 and b_2 .

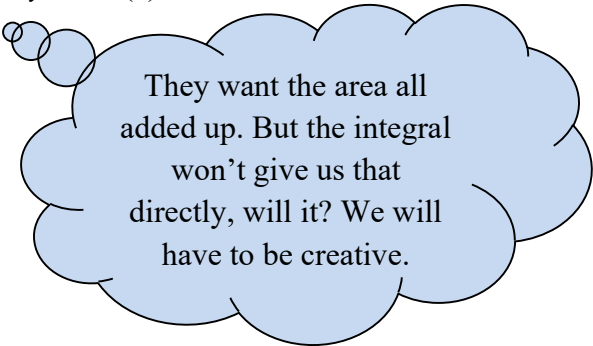
expl 2: Use FTC to find the following.

a.) $\int_1^9 \frac{2}{\sqrt{x}} dx$

Think
antiderivatives
now.

b.) $\int_0^4 3t(t+2) dt$

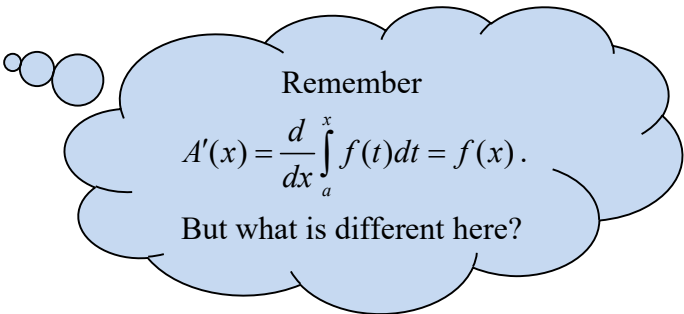
expl 3: Find the area of the region bounded by the graph of $y = \cos(x)$ and the x -axis on the interval $\left[\frac{\pi}{4}, \pi\right]$. Start with a graph.



They want the area all added up. But the integral won't give us that directly, will it? We will have to be creative.

expl 4: Simplify the expression.

$$\frac{d}{dx} \int_x^4 (t^3 + 5) dt$$

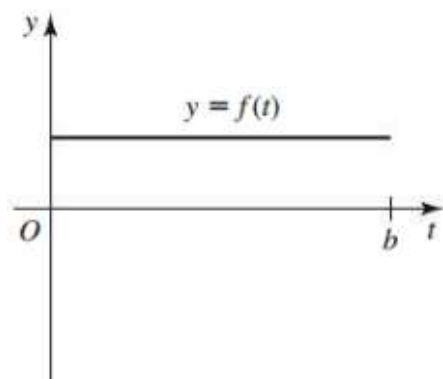


Remember

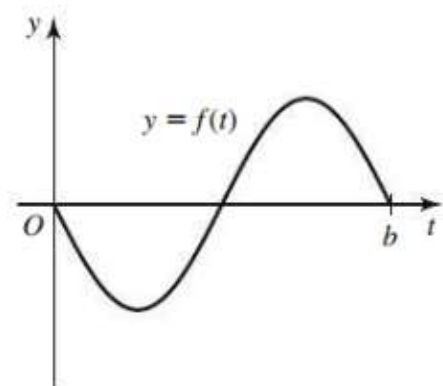
$$A'(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x).$$

But what is different here?

expl 5a: Analyze the graph of $f(t)$ to the right to draw a possible graph of $A(x) = \int_0^x f(t)dt$ on top of this graph.



expl 5b: Analyze the graph of $f(t)$ to the right to draw a possible graph of $A(x) = \int_0^x f(t)dt$ on top of this graph.



expl 6: Use FTC, Part 1, to find the function f that satisfies the equation below. Verify the result by substitution into the equation.

$$\int_0^x f(t)dt = 4 \tan x + 5x^2 - 4$$

