We need a solid, mathematically defined concept for limit. We may see an even more rigorous definition later.

Calculus I

Class notes

Definitions of Limits (section 2.2)

A later section (which we may cover) will add rigor to this concept. For now, this will do. We will define the limit of a function at a certain value of x (independent variable).

## **Definition: Limit of a Function:**

Suppose a function f is defined for all values of x near x = a except possibly at a.

That sounds a little crazy but what we want to picture is a function like this one here. Notice it is defined (shown by the points of the graph) for all values except x = 3. So, think of a as any input, even 3.

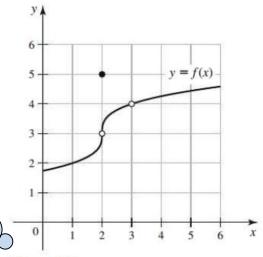


Figure 2.7

If f(x) is arbitrarily close to L (meaning as close to L as we would like) for all x sufficiently close (but *not* equal) to a, we write  $\lim_{x \to a} f(x) = L$ .

This will make more sense as you practice.

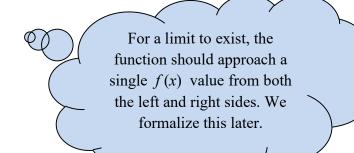
Pronounced "the limit, as x approaches a, of f(x)".

expl 1: Consider the graph given above. Find the following.

a.) 
$$f(1)$$
 and  $\lim_{x\to 1} f(x)$ 

b.) 
$$f(2)$$
 and  $\lim_{x\to 2} f(x)$ 

c.) 
$$f(3)$$
 and  $\lim_{x\to 3} f(x)$ 



# Finding Limits from a Table of Values:

expl 2: Calculate the following values for  $f(x) = \frac{x^2 - 4}{x - 2}$  and use it to conjecture about the value

of  $\lim_{x\to 2} f(x)$ .

X	1.9	1.99	1.999	1.9999
f(x)				

A little algebra will help us out. Factor the top and simplify.

As x approaches 2 from the left, what does the table indicate f(x) is approaching? We will write this as  $\lim_{x \to a} f(x)$ . Practice the notation here.

Do you see the itty bitty "negative sign"

to the right of the 2?

x	2.1	2.01	2.001	2.0001
f(x)				

As x approaches 2 from the right, what does the table indicate f(x) is approaching? We will write this as  $\lim_{x \to x^+} f(x)$ . Practice the notation here.

Do you see the itty bitty "positive sign" to the right of the 2?

As we will see in the next theorem, if these two (one-sided) limits agree, then the limit of f(x) as x approaches 2 is that same number. Practice the notation by writing that now.

Let's fill in some details with formal definitions.

### **One-sided Limits:**

## **Definition: Right-sided Limit:**

Suppose a function f is defined for all values of x near x = a with x > a. If f(x) is arbitrarily close to L for all x sufficiently close to a with x > a, we write  $\lim_{x \to a} f(x) = L$ .

Pronounced "the limit, as x approaches a from the right, of f(x) is L"

#### **Definition: Left-sided Limit:**

Suppose a function f is defined for all values of x near x = a with x < a. If f(x) is arbitrarily close to L for all x sufficiently close to a with x < a, we write  $\lim_{x \to a} f(x) = L$ .

Pronounced "the limit, as x approaches a from the left, of f(x) is L"

As you might have guessed from the last example, we can deduce the limit of f(x) as x approaches a if we know both of these one-sided limits. Here's a theorem.

## Theorem 2.1: Relationship between one-sided and two-sided limits:

Suppose a function f is defined for all values of x near x = a except possibly at a. Then  $\lim_{x \to a} f(x) = L$  if and only if  $\lim_{x \to a} f(x) = L$ .

# The Language of Logic:

"If p, then q" means if p is true, then you know q is also true. The phrase "p if and only if q" means both "If p, then q" and, at the same time, "If q, then p". This is equivalent to saying that p and q are the same statement.

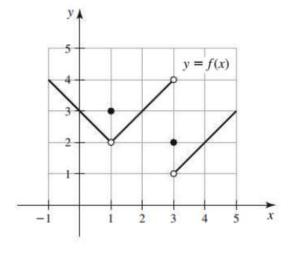
So, to find a limit such as  $\lim_{x\to 2} f(x)$ , we must find both  $\lim_{x\to 2^+} f(x)$  and  $\lim_{x\to 2^-} f(x)$ . If they agree, then the limit "as x approaches 2" exists and is this common number.

expl 3: Find the following using the graph to the right.

a.) f(3)



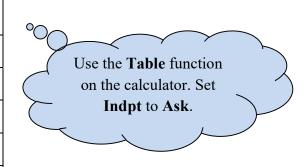
- $c.) \lim_{x \to 3^+} f(x)$
- $d.) \lim_{x \to 3} f(x)$



Graphs and table can be deceiving or hard to interpret. Be you careful!

expl 4: Create a table of values for  $y = \tan\left(\frac{3}{x}\right)$  as started below. Describe the pattern. What happens to  $y = \tan\left(\frac{3}{x}\right)$  as x approaches 0? In other words, what is  $\lim_{x\to 0} \tan\left(\frac{3}{x}\right)$ ?

x	$y = \tan\left(\frac{3}{x}\right)$
$12/\pi$	
$\frac{12}{3\pi}$	
$12/5\pi$	
$12/7\pi$	
$\frac{12}{9\pi}$	



What the heck is going on here? It does *not* look as though the function is approaching any one value as *x* approaches 0?!? Does graphing the function help?

Try the Standard window. Then Zoom In more and more. Does that help?

Eek! It turns out that this limit does *not* exist. It is hard to see that from the graph. We will need other analytical methods to decipher some limits. More on that later.